Chapter 4

Matrices and Determinants

Matrices →

A system of mn numbers (real and complex) arranged in a rectangular array of m rows and n columns is called a matrices of order $m \times n$ or an $m \times n$ order matrix.

A $m \times n$ matrix is written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

where a_{ij} represent the number in the ith row and jth column. The number a_{11}, a_{12}, a_{13} etc are called the elements or entries of the matrix.

Three different systems of enclosing the numbers constituting a matrix are as follows

[ ], ( ), || ||.

we shall use [ ], in this book to represent a matrix.

Type of Matrix

(1) Row Matrix $\longrightarrow$ A single row matrix is called a row matrix or a row vector.

eg: the matrix $[1, 3, 5]$ is 1×3 row matrix.

(2) Column Matrix $\longrightarrow$ A single column matrix is called a column matrix or a column ~~matrix~~ vector.

eg. the matrix $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ is a 3×1 column matrix.

(3) Square Matrix $\longrightarrow$ If $m=n$ i.e. if the number of rows and column of a matrix are equal say n, then it is called a square matrix of order n.

(4) Null or zero matrix $\longrightarrow$ If all the element of a matrix are equal to zero, than it is called a null matrix and it is denoted by O.

(5) <u>Diagonal matrix</u> → A square matrix in which all its elements are zero except those in the leading diagonal is called a diagonal matrix.

The diagonal matrices of order 2 and 3 are as follows

$$\begin{bmatrix} K_1 & 0 \\ 0 & K_1 \end{bmatrix}, \quad \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{bmatrix}$$

(6) <u>Scalar matrix</u> → A square matrix in which all the diagonal elements are equal and all other elements equal to zero is called a scalar matrix.

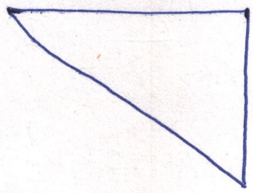
i.e $\begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{bmatrix}$ is a scalar matrix.

(7) <u>Unit matrix or Identity matrix</u> → A square matrix in which all its diagonal elements are equal to 1 and all other elements equal to zero is called a unit matrix or Identity matrix.

eg. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

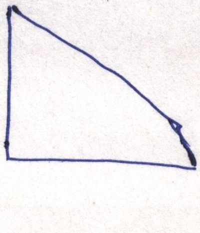
(8) Upper triangular matrix → A square matrix A whose elements $a_{ij} = 0$ for $i > j$ is called an upper triangular matrix.

eg. $A = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 7 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ is an upper triangular matrix



(9) Lower triangular matrix → A square matrix A whose elements $a_{ij} = 0$ for $i < j$ is called a lower triangular matrix.

eg. $A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 7 & 0 \\ 6 & 2 & 1 \end{bmatrix}$ is a lower triangular matrix.



Equal matrix →

Two matrix A and B are said to be equal, written $A = B$ if

(i) they are both of the same order i.e. have the same number of rows and column, and

(ii) the elements in the corresponding places of the two matrix are the same.

Transpose of a matrix →

Let A be a $m \times n$ matrix. Then the matrix of order $n \times m$ obtained by changing its rows in columns and columns into rows is called the transpose of A and is denoted by A' or A^T or A^t.

Note:- The transpose of $m \times n$ matrix is $n \times m$ matrix.

e.g. Let $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 4 & 2 & 6 & 4 \end{bmatrix}_{2 \times 4}$

then $A' = \begin{bmatrix} 1 & 4 \\ 0 & 2 \\ 2 & 6 \\ 3 & 4 \end{bmatrix}_{4 \times 2}$

Some important results of transposed matrix

(i) $(A+B)' = A' + B'$

(ii) $(KA)' = KA'$, when K is constant.

(iii) $(AB)' = B'A'$

Negative of a matrix →

Let A be a matrix. Then the negative of the matrix A is defined as the matrix $-A$.

Symmetric matrix →

A square matrix A is said to be symmetric if $A' = A$.

For eg.
if $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

then $A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

Here $\boxed{A' = A}$

Hence A is a symmetric Matrices.

Skew Symmetric Matrix →

A square matrix A is said to be skew symmetric matrix if $A' = -A$

For eg.

If
$$A = \begin{bmatrix} 0 & -4 & -3 \\ 4 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix}$$

then
$$A' = \begin{bmatrix} 0 & 4 & 3 \\ -4 & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -4 & -3 \\ 4 & 0 & -1 \\ -3 & -1 & 0 \end{bmatrix} = -A$$

Hence $A' = -A$

Hence A is a ~~symmetric~~ skew symmetric matrix.

[यहाँ पर पेज नं 10 से मिलाना है]

Addition of Matrix.

Let there be two matrices ~~A~~ A and B of the same order $m \times n$. Then their sum denoted $A+B$ is defined to be the matrix of order $m \times n$ obtained by adding the corresponding elements of A and B.

For eg. $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 2 \\ 1 & 0 \end{bmatrix}$

then $A+B = \begin{bmatrix} 9 & 5 \\ 2 & 5 \end{bmatrix}$

Note

we can find the sum $A+B$ only if the matrix A and B are of the same order i.e. if the number of rows and columns of matrix A and B are equal.

Scalar multiplication of a matrix

Let A be a matrix and k a scalar. Then the matrix obtained by multiplying each element of matrix A by k is called the scalar multiple of A by k and is denoted by kA or Ak

Ex $A = \begin{bmatrix} 1 & 3 \\ 5 & 0 \end{bmatrix}$

then $3A = \begin{bmatrix} 3.1 & 3.3 \\ 5.3 & 0.3 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 15 & 0 \end{bmatrix}$

Multiplication of matrix

Let A be $m \times n$ and B be $n \times p$ matrices. Then the product of matrices A and B denoted by AB is the matrix of order $m \times p$.

Ex. If $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 3 \\ 2 & -1 & 0 \\ 1 & 3 & 2 \end{bmatrix}$

Then

$$AB = \begin{bmatrix} 1\times -1 + 2\times 2 + 1\times 1 & 1\times 2 + 2\times -1 + 1\times 3 & 1\times 3 + 2\times 0 + 2\times 2 \\ 2\times -1 + 3\times 2 + 4\times 1 & 2\times 2 + 3\times -1 + 4\times 3 & 2\times 3 + 3\times 0 + 4\times 2 \\ 1\times -1 + 2\times 2 + 3\times 1 & 1\times 2 + 2\times -1 + 3\times 3 & 1\times 3 + 2\times 0 + 3\times 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 3 & 7 \\ 8 & 13 & 14 \\ 6 & 9 & 9 \end{bmatrix}$$

Note → $AB \neq BA$.

Note

Every square matrix can be expressed as the sum of a symmetric and skew symmetric matrix. i.e

$$\boxed{A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')}$$

where $\frac{A+A'}{2}$ is a symmetric matrix and

$\frac{A-A'}{2}$ is a skew symmetric matrix.

Example:- Construct a 2×2 matrix, whose elements are $a_{ij} = \frac{(i+2j)^2}{2}$

Sol:- Here $a_{ij} = \frac{(i+2j)^2}{2}$

$$a_{11} = \frac{(1+2)^2}{2} = \frac{9}{2} \text{ and } a_{12} = \frac{(1+4)^2}{2} = \frac{25}{2}$$

$$a_{21} = \frac{(2+2)^2}{2} = 8 \text{ and } a_{22} = \frac{(2+4)^2}{2} = 18$$

So Required matrix is $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 9/2 & 25/2 \\ 8 & 18 \end{bmatrix}$

Example 2 – Construct a 3×3 matrix whose elements are given by

(i) $a_{ij} = i \times j$ (ii) $a_{ij} = 2i - 3j$

Sol a_{ij} denotes the elements of a matrix which lie in the ith row and jth column

A 3×3 matrix, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(i)
Given $a_{ij} = i \times j$

$\therefore$ $a_{11} = 1 \times 1 = 1$, $a_{12} = 1 \times 2 = 2$, $a_{13} = 1 \times 3 = 3$

$a_{21} = 2 \times 1 = 2$, $a_{22} = 2 \times 2 = 2$, $a_{23} = 2 \times 3 = 6$

$a_{31} = 3 \times 1 = 3$, $a_{32} = 3 \times 2 = 6$, $a_{33} = 3 \times 3 = 9$

$\therefore$ Required matrix $= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

(ii) Given $a_{ij} = 2i - 3j$

$\therefore$ $a_{11} = 2 \times 1 - 3 \times 1 = 2 - 3 = -1$

$a_{12} = 2 \times 1 - 3 \times 2 = 2 - 6 = -4$

$a_{13} = 2 \times 1 - 3 \times 3 = 2 - 9 = -7$

$a_{21} = 2 \times 2 - 3 \times 1 = 4 - 3 = 1$

$a_{22} = 2\times2 - 3\times2 = 4-6 = -2$

$a_{23} = 2\times2 - 3\times3 = 4-9 = -5$

$a_{31} = 2\times3 - 3\times1 = 6-3 = 3$

$a_{32} = 2\times3 - 3\times2 = 6-6 = 0$

$a_{33} = 2\times3 - 3\times3 = 6-9 = -3$

$\therefore$ Required matrix $= \begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \\ 3 & 0 & -3 \end{bmatrix}$

Example 3.- If $A = \begin{bmatrix} x & 3x-y \\ 2x+z & 3y-w \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$,

find x, y, z, w.

Sol :- Now comparing, the corresponding elements of the two matrices, we get

$x = 3$

$3x-y = 2 \Rightarrow 3\times3-y = 2 \Rightarrow y = 9-2 = 7$

$2x+z = 4 \Rightarrow 2\times3+z = 4 \Rightarrow z = 4-6 = -2$

and $3y-w = 7 \Rightarrow 3\times7-w = 7 \Rightarrow w = 21-7 = 14$

Hence $x = 3$, $y = 7$, $z = -2$, and $w = 14$

Example 4 Find X and Y, if $X+Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and

$$X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Sol :-

$$X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} \quad \text{——— (1)}$$

$$X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{——— (2)}$$

Adding (1) + (2), we get

$$2X = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$2X = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 10/2 & 0/2 \\ 2/2 & 8/2 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

Subtracting (1) ~~and~~ (2), we get

$$2Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$2Y = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$$

$$Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Example 5 If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find k so that $A^2 = 8A + kI$

Sol, we have ~~~~ $A^2 = 8A + kI$

so $$A^2 = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}$$

$$8A = 8\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix}$$

$$kI = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Then $$A^2 = 8A + KI$$

$$\begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$$

Now Comparing, the corresponding elements of the two matrices we get,

$8+K = \phi$ or $56+K = 49$

$\Rightarrow \boxed{K = -7}$

Example 6 Find x, if $\begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = 0$

Sol we have $\begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} x\times 1 + 1\times -2 & x\times 0 + 1\times -3 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} x-2 & -3 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} x(x-2) + -3\times 3 \end{bmatrix} = 0$

$\Rightarrow x^2 - 2x - 9 = 0$

$\Rightarrow x = \dfrac{2 \pm \sqrt{4+36}}{2} = \dfrac{2 \pm 2\sqrt{10}}{2}$

Hence $x = 1 \pm \sqrt{10}$

Example: 7 If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that

$$A^2 = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

Sol $A^2 = A \cdot A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$$= \begin{bmatrix} \cos\cdot\cos\alpha - \sin\alpha\sin\alpha & \sin\alpha\cos\alpha + \cos\alpha\sin\alpha \\ -\sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \cos\alpha\cdot\cos\alpha - \sin\alpha\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & \sin(\alpha+\alpha) \\ -\sin(\alpha+\alpha) & \cos^2\alpha - \sin^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

Example 8 If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, find $A^2 - 5A + 4$

Sol $A^2 = A \cdot A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

Now, $A^2 - 5A + 4$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5 \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} + \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5-10+4 & -1+0+0 & 2-5+0 \\ 9-10+0 & -2-5+4 & 5-15+0 \\ 0-5+0 & -1+5+0 & -2+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix}$$

Example 9 Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $f(x) = x^2 - 4x + 7$, show that $f(A) = 0$

Sol :-

$$f(A) = A^2 - 4A + 7$$

$$\therefore \quad A^2 = A \cdot A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix}.$$

$$-4A = -4 \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -8 & -12 \\ 4 & -8 \end{bmatrix}$$

and $7I = 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$$\therefore \quad A^2 - 4A + 7 = f(A)$$

$$\therefore \quad f(A) = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} + \begin{bmatrix} -8 & -12 \\ 4 & -8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -12 \\ 4 & -1 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence $f(A) = 0$.

Example 10 Express the following matrices as the sum of symmetric and skew-symmetric matrices.

$$\begin{bmatrix} 6 & 1 \\ 3 & 4 \end{bmatrix}$$

Sol: Here, $A = \begin{bmatrix} 6 & 1 \\ 3 & 4 \end{bmatrix}$

$$A' = \begin{bmatrix} 6 & 3 \\ 1 & 4 \end{bmatrix}$$

We know that $A = \left(\frac{A+A'}{2}\right) + \left(\frac{A-A'}{2}\right)$ ——— (1)

where $\frac{A+A'}{2}$ is symmetric and $\left(\frac{A-A'}{2}\right)$ is skew-symmetric

$$\therefore \frac{A+A'}{2} = \frac{1}{2}\left\{\begin{bmatrix} 6 & 1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 6 & 3 \\ 1 & 4 \end{bmatrix}\right\}$$

$$\frac{A+A'}{2} = \frac{1}{2}\begin{bmatrix} 12 & 4 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$$

$$A - A' = \begin{bmatrix} 6 & 1 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 3 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$\therefore \frac{A - A'}{2} = \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\therefore$ Putting the values of A, $\frac{A+A'}{2}$ and $\frac{A-A'}{2}$ in eq (1), we get

$$\begin{bmatrix} 6 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

DETERMINANTS

Any number associated with a square matrix is called the determinant of the matrix.

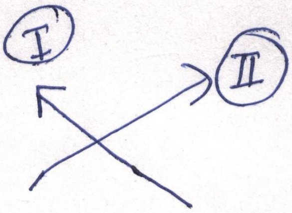
Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3.

Then determinant of A is denoted by $|A|$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

To find the value of determinant of order 2.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$



then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$$= (a_{11} a_{22} - a_{21} a_{12})$$

For example if $|A| = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 6 - 3 = 3$.

To find the value of determinant of order 3.

Let $|A| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 5 & 1 \\ 2 & 1 & 0 \end{vmatrix}$

$$|A| = 2\begin{vmatrix} 5 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} + 3\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix}$$

$|A| = 2(0-1) - 1(0-2) + 3(1-10)$

$|A| = -2 + 2 - 27$

$|A| = -27$

Example

Minors :-

The minor of an element of determinant is the determinant obtained by deleting the row and column of the determinant in which that element exists

For ex :- $|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 5 & 1 \\ 3 & 1 & 7 \end{vmatrix}$

$$\text{Minor of } 5 = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 5 & 1 \\ 3 & 1 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} = 7 - 6 = 1$$

Similarly minors of other elements can be calculated.

Co-factors :-

The minor of the element in the ith row and jth column of a determinant multiplied by $(-1)^{i+j}$ is called the co-factor of the element.

In the previous example,

Cofactor of element 4 will be $= (-1)^{2+2}(1)$

$= (-1)^4 (1) = 1$.

Laplace Expansion Method:-

Example:-11 Expand by Laplace expansion. $\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$

Sol:- $\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$

Expand by Ist row

$$= 1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= 1(1-4) - 2(2-4) + 2(4-2)$$

$$= -3 + 4 + 4 = 5$$

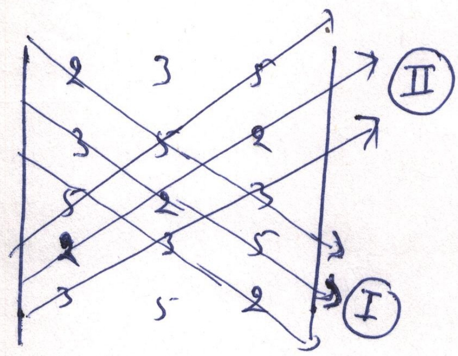
Sarrus rule :-

Example 12

Evaluate the determinant by Sarrus Rule.

$$\begin{vmatrix} 2 & 3 & 5 \\ 3 & 5 & 2 \\ 5 & 2 & 3 \end{vmatrix}$$

Sol : We have



$$= (2\times5\times3) + (2\times3\times5) + (5\times3\times2) - (5\times5\times5) - (2\times2\times2) - (3\times3\times3)$$

$$= 30 + 30 + 30 - 125 - 8 - 27$$

$$= 90 - 160 = \underline{-70}.$$

Singular and non Singular matrix :-

A Square matrix is called Singular matrix if its determinant is zero otherwise A is non singular matrix. i.e $|A| = 0$ Singular, $|A| \neq 0$, non singular.

Properties of determinants :—

(1) The value of a determinant is not changed when rows and column are interchanged.

$$\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

(2) If two rows (or columns) of a determinant are interchanged, then the sign of value of the determint is multiplied by (-1)

$$\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \text{ and } \Delta' = \begin{vmatrix} p & q & r \\ a & b & c \\ x & y & z \end{vmatrix},$$

then $\boxed{\Delta = -\Delta'}$

(3) If two rows (or columns) of a determinant are identical, then the value of determinant is zero.

$$\begin{vmatrix} a & b & c \\ p & q & r \\ a & b & c \end{vmatrix} = 0$$

(4) If the elements of a rows (or a column) of a determinant are multiplied by a scalar, then the value of the new determinant is equal to same scalar times the value of the original determinant.

$$\begin{vmatrix} k a & kb & kc \\ p & q & r \\ x & y & z \end{vmatrix} = k \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

(5) If elements of any row or column of a determinant is sum (difference) of two or more elements, then the determinant can be expressed as sum (difference) of two or more determinant.

$$\begin{vmatrix} a+\alpha & p & x \\ b+\beta & q & y \\ c+\gamma & r & z \end{vmatrix} = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} + \begin{vmatrix} \alpha & p & x \\ \beta & q & y \\ \gamma & r & z \end{vmatrix}$$

Example 13 without expansion show that each of the following determinant vanishes.

(i) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ (ii) $\begin{vmatrix} 9 & 9 & 12 \\ 1 & -3 & -4 \\ 1 & 9 & 12 \end{vmatrix}$

(iii) $\begin{vmatrix} -5 & 3 & 10 \\ 7 & 6 & -14 \\ 9 & 8 & -18 \end{vmatrix}$

Sol:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Oprateing $C_2 - C_1$ and $C_3 - C_2$, we have

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 1 \\ 7 & 1 & 1 \end{vmatrix}$$

Column two and three are identical so

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 1 \\ 7 & 1 & 1 \end{vmatrix} = 0$$

(ii) $\begin{vmatrix} 9 & 9 & 12 \\ 1 & -3 & -4 \\ 1 & 9 & 12 \end{vmatrix}$

Taking ~~out~~ 3 common from second column and 4 third column

So $3 \times 4 \begin{vmatrix} 9 & 3 & 3 \\ 1 & -1 & -1 \\ 1 & 3 & 3 \end{vmatrix}$

Column two and three are identical so

$$\begin{vmatrix} 9 & 3 & 3 \\ 1 & -1 & -1 \\ 1 & 3 & 3 \end{vmatrix} = 0$$

(iii) $\begin{vmatrix} -5 & 3 & 10 \\ 7 & 6 & -14 \\ 9 & 8 & -18 \end{vmatrix}$

Taking out 2 common from third column, we get

$$\begin{vmatrix} -5 & 3 & 5 \\ 7 & 6 & -7 \\ 9 & 8 & 9 \end{vmatrix}$$

Column Ist and third are identical so its value is zero.

Example 14 Show that the following matrix are Singular. (i) $\begin{vmatrix} 9 & 27 \\ 2 & 6 \end{vmatrix}$ (ii) $\begin{vmatrix} -5 & 2 \\ 1 & 0 \end{vmatrix}$

Sol: (i) $A = \begin{vmatrix} 9 & 27 \\ 2 & 6 \end{vmatrix} = 54 - 54 = 0$

$|A| = 0$

$\Rightarrow$ A is singular

(ii) $A = \begin{vmatrix} -5 & 2 \\ 1 & 0 \end{vmatrix} \Rightarrow |A| = 0 - 2 = -2$

$|A| \neq 2$

$\Rightarrow$ A is non singular.

Example 15 with out expansion find the value of

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

Sol: Here ~~Let~~ $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

Operating $C_3 \to C_3 + C_2$, we get

$$\begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$

C_1 and C_3 are identical so

$$(a+b+c)(0) = \underline{0}.$$

Example: 16 Find the value of $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

where ω is cube root of unity.

Sol: $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

Operation: $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega^2 & 1 \\ 1+\omega+\omega^2 & 1 & \omega \end{vmatrix} = \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & \omega^2 & 1 \\ 0 & 1 & \omega \end{vmatrix} \quad (1+\omega+\omega^2=0)$$

$= 0$ ($\because$ all the elements of C_1 are zero)

Example 17 Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc+bc+ac+ab$$

Sol:- L.H.S $= \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

Operating $C_1 \to \frac{C_1}{a}$, $C_2 \to \frac{C_2}{b}$, $C_3 \to \frac{C_3}{c}$

$$= abc \begin{vmatrix} 1+\frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & 1+\frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b} & 1+\frac{1}{c} \end{vmatrix}$$

Operating $C_1 \to C_1+C_2+C_3$, we get

$$= abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & 1+\frac{1}{c} \end{vmatrix}$$

$$= (abc)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 1 & 1+\frac{1}{b} & \frac{1}{c} \\ 1 & \frac{1}{b} & 1+\frac{1}{c} \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= (abc)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (abc)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(1)$$

$$= (abc)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$= abc + \frac{abc}{a} + \frac{abc}{b} + \frac{abc}{c}$$

$$= abc + bc + ac + ab.$$

$= R.H.S$

$\boxed{L.H.S = R.H.S}$

Example 18 Evaluate $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$.

Sol: $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta$

$= 1$.

Example 19 Evaluate $\begin{vmatrix} x+d & x & x \\ x & x+d & x \\ x & x & x+d \end{vmatrix}$

Sol: Operating $C_1 \to C_1 + C_2 + C_3$, we have

$$\begin{vmatrix} 3x+d & x & x \\ 3x+d & x+d & x \\ 3x+d & x & x+d \end{vmatrix}$$

$$= (3x+d)\begin{vmatrix} 1 & x & x \\ 1 & x+d & x \\ 1 & x & x+d \end{vmatrix}$$

Operating $R_2 - R_1$ and $R_3 - R_2$, we get

$$= (3x+d)\begin{vmatrix} 1 & x & x \\ 1 & d & 0 \\ 0 & 0 & d \end{vmatrix}$$

$= (3x+d)(1 \cancel{*} d + d)$

$= d^2(3x+d)$

Example: 20 Prove that $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

Sol: Given that, $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$

Taking a, b, c common from R_1, R_2 and R_3 respectively, we have

$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$

Again taking a, b, c common from C_1, C_2 and C_3 respectively, we have

$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

Operating $R_1 \to R_1 + R_2$, we have

$$= a^2 b^2 c^2 \begin{vmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Expanding in terms of R_1, we get

$$= a^2 b^2 c^2 (2)(1+1)$$

$$= 4 a^2 b^2 c^2$$

Example 21 Solve the equation

$$\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

Sol :-

$$\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

Operating $R_1 \to R_1 + R_2 + R_3$, we have

$$\begin{vmatrix} 3x-2 & 3x-2 & 3x-2 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

$$(3x-2)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

Operating $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$, we have,

$$(3x-2)\begin{vmatrix} 1 & 0 & 0 \\ 3 & 3x-11 & 11-3x \\ 3 & 0 & 3x-11 \end{vmatrix} = 0$$

Expanding in terms of R_1, we get

$$(3x-2)\left[3x-11\right]^2 = 0$$

$$\Rightarrow \quad 3x-2 = 0 \quad \text{or} \quad 3x-11 = 0$$

$$\Rightarrow \quad \boxed{x = 2/3 \quad \text{or} \quad x = 11/3\,.}$$

Example 22 Evaluate the determinant using Pivotal method

$$\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 7 & 6 & 1 \end{vmatrix}$$

Sol: – Let $|A| = \begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 7 & 6 & 1 \end{vmatrix}$

Using this method, we have

$$|A| = \frac{1}{(2)^{3-2}} \begin{vmatrix} \begin{vmatrix} 2 & -4 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & -4 \\ 7 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 7 & 1 \end{vmatrix} \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 14 & -5 \\ 40 & -19 \end{vmatrix}$$

$$= \frac{1}{2}\left[14 \times -19 - (-5) \times 40\right]$$

$$= \frac{1}{2}\left[-266 + 200\right]$$

$$= -33.$$

~~we have~~

Example 23

Prove that $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

Sol: $L.H.S = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$

Operating $C_1 \to C_1 + C_2 + C_3$, we have

$$\begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(p+q+r) & r+p & p+q \\ 2(x+y+z) & z+x & x+y \end{vmatrix}$$

Operating $C_1 \to C_1 - C_2$

$$= 2\begin{vmatrix} b & c+a & a+b \\ q & r+p & p+q \\ y & z+x & x+y \end{vmatrix}$$

Operating $C_3 \to C_3 - C_1$, we have

$$= 2\begin{vmatrix} b & c+a & a \\ q & r+p & p \\ y & z+x & x \end{vmatrix}$$

Operating $C_2 \to C_2 - C_3$, we got.

$$= 2\begin{vmatrix} b & c & a \\ q & r & p \\ y & z & x \end{vmatrix}$$

$$= -2\begin{vmatrix} a & c & b \\ p & r & q \\ x & z & y \end{vmatrix} \quad (\text{Interchange } C_1 \text{ and } C_3)$$

$$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \quad (\text{Again interchange } C_2 \text{ and } C_3)$$

$= \underline{R.H.S}$.

<u>Example</u>: 24 Show that $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

<u>Sol</u>: — L.H.S $= \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

multiply R_1 by x, R_2 by y and R_3 by z, we have

$$= xyz\begin{vmatrix} x & x^2 & xyz \\ y & y^2 & xyz \\ z & z^2 & xyz \end{vmatrix}$$

$$= \frac{xyz}{xyz}\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix} \quad (\text{Interchanging } C_1 \text{ and } C_3)$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad (\text{Interchanging } C_2 \text{ and } C_3)$$

= R.H.S.

<u>Solution of Simultaneous linear eqs by Cramer's rule</u>

Consider the following eqns

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

By Cramer's Rule.

$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}, \quad z = \frac{D_3}{D}, \quad D \neq 0$$

Note:-

(i) Consistancy of simultaneous eq. If $D \neq 0$, the eq^n is consistant and has unique solution by Cramer's rule

(ii) If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then given system of eq is consistant with ∞ many solution.

(iii) If $D = 0$, and atleast one of the D_1, D_2 and D_3 is non zero, then the given system of eq is inconsistant.

Example: 25 Using determinants solve $x+2y=3$
$4x+y=5$

Sol: we have

$$D = \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} = 1\times1-4\times2 = 1-8 = -7$$

$$D_1 = \begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix} = 3\times1-2\times5 = 3-10 = -7$$

or $$D_2 = \begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix} = 5\times1-3\times4 = 5-12 = -7$$

$\therefore$ By Cramer's rule we have

$$x = \frac{D_1}{D} = \frac{-7}{-7} = 1$$

$$y = \frac{D_2}{D} = \frac{-7}{-7} = 1$$

Hence $x=1$, $y=1$ is the required solution.

Example 26

Using Cramer's rule solve

$$6x + y - 3z = 5$$
$$x + 3y - 2z = 5$$
$$2x + y + 4z = 8$$

Sol: we have

$$D = \begin{vmatrix} 6 & 1 & -3 \\ 1 & 3 & -2 \\ 2 & 1 & 4 \end{vmatrix}$$

$$D = 6(12+2) - 1(4+4) - 3(1-6)$$

$$D = 84 - 8 + 15 = 91$$

$$\boxed{D = 91}$$

$$D_1 = \begin{vmatrix} 5 & 1 & -3 \\ 5 & 3 & -2 \\ 8 & 1 & 4 \end{vmatrix} = 5(12+2) - 1(20+16) - 3(5-24)$$

$$= 70 - 36 + 57 = 91$$

$$\boxed{D_1 = 91}$$

$$D_2 = \begin{vmatrix} 6 & 5 & -3 \\ 1 & 5 & -2 \\ 2 & 8 & 4 \end{vmatrix} = 6(20+16) - 5(4+4) - 3(8-10)$$

$$= 216 - 40 + 6 = 182$$

$$\boxed{D_2 = 182}$$

$$D_3 = \begin{vmatrix} 6 & 1 & 5 \\ 1 & 3 & 5 \\ 2 & 1 & 8 \end{vmatrix}$$

$$= 6(24-5) - 1(8-10) + 5(1-6)$$

$$= 114 + 2 - 25 = 91$$

$$\boxed{D_3 = 91}$$

$\therefore$ By Cramer's rule, we have

$$x = \frac{D_1}{D} = \frac{91}{91} = 1$$

$$y = \frac{D_2}{D} = \frac{182}{91} = 2$$

$$z = \frac{D_3}{D} = \frac{91}{91} = 1$$

Hence $x=1$, $y=2$, $z=1$ is the required solution.

Example 27. Classify the following system of eq^n as consistent or inconsistent. ~~If consistent, then soln.~~

(i) $3x - 2y = 4$
$6x - 4y = 10$

(ii) $x + 2y = 5$
$3x + 6y = 15$

Sol:- (i) We have $D = \begin{vmatrix} 3 & -2 \\ 6 & -4 \end{vmatrix} = -12 + 12 = 0$

$$\boxed{D = 0}$$

$$D_1 = \begin{vmatrix} 4 & -2 \\ 10 & 4 \end{vmatrix} = 16 + 20 = 36$$

$$\boxed{D_1 = 36}$$

Here $D = 0$, $D_1 \neq 0$

Hence, the given system of eqs is inconsistent.

(ii) We have, $D = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6 - 6 = 0$

$D = 0$

$$D_1 = \begin{vmatrix} 5 & 2 \\ 15 & 6 \end{vmatrix} = 30 - 30 = 0$$

$$\boxed{D_1 = 0}$$

$$D_2 = \begin{vmatrix} 1 & 5 \\ 3 & 15 \end{vmatrix} = 15 - 15 = 0$$

$$\boxed{D_2 = 0}$$

Here $D = 0$, $D_1 = 0$, $D_2 = 0$

So, the given system of eqns has infinitely many solution.

Adjoint of A matrix

If A is a square matrix, then the adjoint of A is defined as the transpose of the matrix obtained by replacing the elements of A by their corresponding cofactors in determinant A.

The adjoint of a square matrix is denoted by adj A.

Inverse of square matrix A.

(i) Find the value of $|A|$

(ii) If $|A| \neq 0$, then only A has its inverse and

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

Example: 28 Find the adjoint of the matrices

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

Sol: Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$. Let C_{ij} be cofactor of a_{ij} in A.

Then, the cofactors of elements of A are given by

$$C_{11} = (+1)\begin{bmatrix} 0 & -2 \\ 0 & 3 \end{bmatrix} = 0$$

$$C_{12} = (-1)\begin{bmatrix} 3 & -2 \\ 1 & 3 \end{bmatrix} = -(9+2) = -11$$

$$C_{13} = (1)\begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix} = 0$$

$$c_{21} = (-1)\begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = -(-3+0) = 3$$

$$c_{22} = (1)\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = (3-2) = 1$$

$$c_{23} = (-1)\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -(0+1) = -1$$

$$c_{31} = (+1)\begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} = (2-0) = 2$$

$$c_{32} = (-1)\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -(-2-6) = 8$$

$$c_{33} = (1)\begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = (0+3) = 3$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 0 & -11 & 0 \\ 3 & 1 & -1 \\ 2 & 8 & 3 \end{bmatrix}' = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

Example 29 Find the inverse of the matrics

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

Sol:- Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, then

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{vmatrix} = 1 \begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} = 1(10) = 10 \neq 0$$

So, A is invertiable. Then the cofactors of elements of A are given by

$$c_{11} = (+1) \begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} = 10$$

$$c_{12} = (-1) \begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} = 0$$

$$c_{13} = (+1) \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0$$

$$c_{21} = (-1) \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = (-1)(10) = -10$$

$$C_{22} = (+1)\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = 5$$

$$C_{23} = (-1)\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{31} = (+1)\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = 8-6 = 2$$

$$C_{32} = (-1)\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = (-1)(4) = -4$$

$$C_{33} = (+1)\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2-0 = 2$$

$$\therefore \text{Adj}\,A = \begin{bmatrix} 10 & 0 & 0 \\ -10 & 5 & 0 \\ 2 & -4 & 2 \end{bmatrix}' = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence $A^{-1} = \dfrac{1}{|A|}\text{Adj}\,A = \dfrac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 1/5 \\ 0 & \frac{1}{2} & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix}$$

Example 30 If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, Show that $A^2 - 5A + 7I = 0$

Hence find A^{-1}.

Sol : We have

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\therefore A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

and
$$A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Now

$$A^2 - 5A + 7I = 0$$

$$\Rightarrow 7I = 5A - A^2$$

$$\Rightarrow I = \frac{5}{7}A - \frac{A^2}{7}$$

$$\Rightarrow A^{-1} = \frac{5}{7}I - \frac{1}{7}A$$

$$\Rightarrow A^{-1} = \frac{5}{7}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{pmatrix} \frac{5}{7} & 0 \\ 0 & 5/7 \end{pmatrix} - \begin{bmatrix} 3/7 & 1/7 \\ -\frac{1}{7} & 2/7 \end{bmatrix}$$

$$= \begin{bmatrix} 5/7 - 3/7 & 0 - 1/7 \\ 0 + \frac{1}{7} & 5/7 - 2/7 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2/7 & -1/7 \\ 1/7 & 3/7 \end{bmatrix}$$

Systems of Simultaneous linear eqs

Matrix Method →

Let us consider three linear eqns in three unknown

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3$$

The system of eqns can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$AX = B, \quad \text{if } |A| \neq 0$$

$$A^{-1}(AX) = A^{-1}B$$

$$\Rightarrow (AA^{-1})X = A^{-1}B$$

$$\Rightarrow IX = A^{-1}B$$

$$\Rightarrow \boxed{X = A^{-1}B}$$

Note: (i) If $|A| \neq 0$, then the system is consistent and has unique solution given by $X = A^{-1}B$

(ii) If $|A| = 0$, and $(\operatorname{adj} A)B = 0$, then the system is consistent with infinite number of solution.

(iii) If $|A| = 0$ and $(\operatorname{adj} A)B \neq 0$, then the system is unconsistant and has no solution.

Example: 31 Solve the system of eqns by matrix method

$$5x + 2y = 4$$
$$7x + 3y = 5$$

Sol: The given system of eqns can be written as

$$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad \text{i.e., } AX = B$$

where $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Now $|A| = \begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix} = 15 - 14 = 1 \neq 0$

$\Rightarrow$ A^{-1} exists and hence the given equation has a unique sol..

Here $C_{11} = (+1)(3) = 3$, $C_{12} = (-1)(7) = -7$

$C_{21} = (-1)(2) = -2$, $C_{22} = (+1)(5) = 5$

$$\therefore \text{adj } A = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}' = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$\text{and } A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

Solution of given system is given by $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 12 & -10 \\ -28 & 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$\therefore x = 2$, $y = -3$.

Example: 32 Solve the following of eqs by matrix method.

$$3x+y+2z=3$$
$$2x-3y-z=-3$$
$$x+2y+z=4$$

Sol:- The given system of eqs can be written as

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

or $\quad AX = B$

where $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

Now

$$|A| = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\begin{vmatrix} -3 & -1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}$$

$$|A| = 8 \neq 0$$

Thus A^{-1} exists hence system has unique solution.

$$X = A^{-1}B$$

Let C_{ij} be the cofactor of a_{ij} in $|A|$

$\Rightarrow C_{11} = -1,\quad C_{12} = -3,\quad C_{13} = 7,\quad C_{21} = 3,\quad C_{22} = 1$

$C_{23} = -5,\quad C_{31} = 5,\quad C_{32} = 7,\quad C_{33} = -11$

$$\text{Adj } A = \begin{bmatrix} -1 & -3 & 7 \\ 3 & 1 & -5 \\ 5 & 7 & -11 \end{bmatrix}' = \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

Hence $$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & 11 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -3-9+20 \\ -9-3+28 \\ 21+15-44 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Here, $x = 1$, $y = 2$, $z = -1$