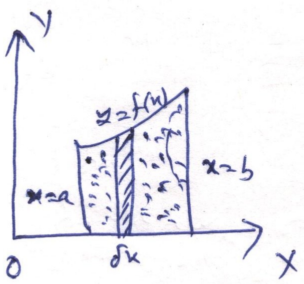
Areas Under Simple Curves

Formulae for computing area of plane curve

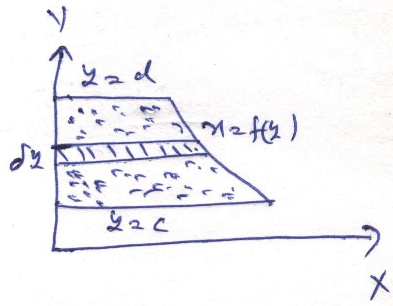
(I) If $f(x)$ is a single valued and continuous function of x in the interval (a, b), then the area (shown by shaded) bounded by the curve $y = f(x)$, the axis of x and the ordinates $x = a$ and $x = b$ is given by

$$\int_a^b y\,dx \quad \text{or} \quad \int_a^b f(x)\,dx$$



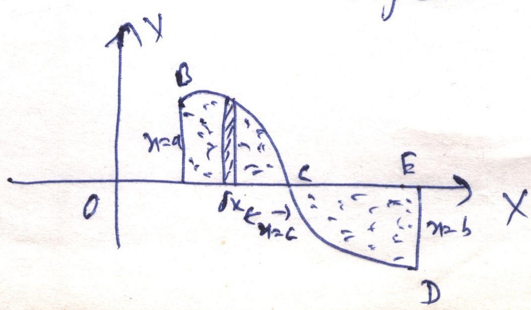
(II) Similarly the area (shown by shaded region) bounded by the curve $x = f(y)$, the axis of y and the abscissa $y = c$ and $y = d$ is given by

$$\int_c^d x\,dy \quad \text{or} \quad \int_c^d f(y)\,dy$$



~~(III) If~~

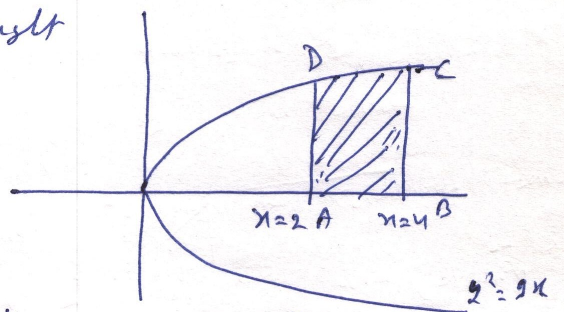
Note: If some part of the curve lies below the axis of x then its area is negative and since area cannot negative, so we consider its magnitude only by taking its modulus.



Example ① Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.

Sol :- The given curve is $y^2 = 9x$, which is parabola with vertex at $(0,0)$ and axis along x-axis.

Here $x = 2$, $x = 4$ are straight lines.



Now

Required Area = Area of ABCD

$$= \int_2^4 y\,dx = \int_2^4 \sqrt{9x}\,dx \cancel{= 3\left[\frac{x^2}{2}\right]_2^4 = 9\left[\frac{4^2}{2} - \frac{2^2}{2}\right]}$$

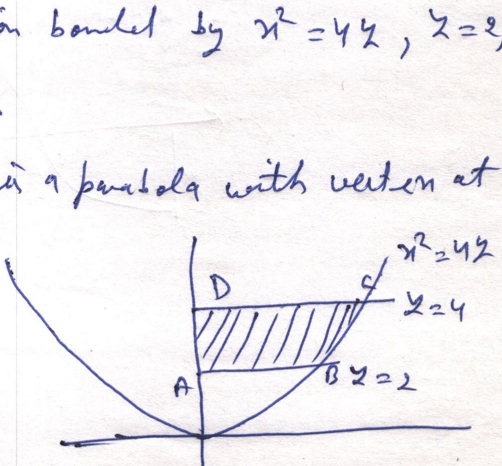
$$= 3\int_2^4 x^{1/2}\,dx$$

$$= 3\left[\frac{x^{3/2}}{3/2}\right]_2^4 = 2\left[4^{3/2} - 2^{3/2}\right]$$

$$= 2\left[8 - 2\sqrt{2}\right] = 16 - 4\sqrt{2}$$ sq. units.

Example 2 Find the Area of region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and y axis in the first quadrant.

Sol :- The given curve $x^2 = 4y$ is a parabola with vertex at $(0,0)$



Required Area = Area of ABCD

$$= \int_2^4 x\,dy$$

$$= \int_2^4 \sqrt{4y}\,dy = 2\int_2^4 y^{1/2}\,dy$$

$$= 2\left[\frac{y^{3/2}}{3/2}\right]_2^4 = \frac{4}{3}\left[4^{3/2} - 2^{3/2}\right] = \frac{4}{3}\left[8 - 2\sqrt{2}\right]$$

$$= \frac{32 - 8\sqrt{2}}{3}$$ sq. units

Exple 3 Find the area of the region bounded by the ellipse.

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

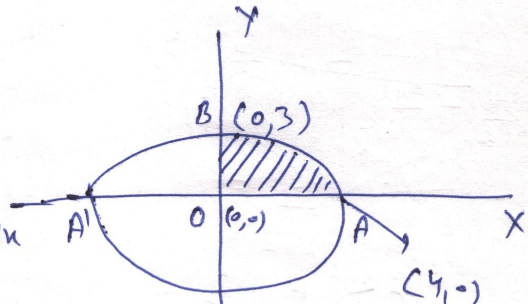
Sol:- Here $\frac{x^2}{16}+\frac{y^2}{9}=1$ is an ellipse

Now $\frac{x^2}{16}+\frac{y^2}{9}=1$

$$\Rightarrow \frac{y^2}{9}=1-\frac{x^2}{16}=\frac{16-x^2}{16}$$

$$\Rightarrow y^2=\frac{9}{16}(16-x^2) \Rightarrow y=\frac{3}{4}\sqrt{16-x^2}$$

Now, Area bounded by the ellipse



$$=4\times(\text{Area of } OABO)$$

$$=4\int_0^4 y\,dx=4\int_0^4 \frac{3}{4}\sqrt{16-x^2}\,dx$$

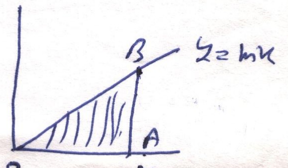
$$=3\int_0^4\sqrt{4^2-x^2}\,dx=3\left[\frac{x}{2}\sqrt{16-x^2}+\frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4$$

$$\left[\because \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$=3\left[\frac{4}{2}\sqrt{16-16}+8\sin^{-1}\left(\frac{4}{4}\right)-0-8\sin^{-1}\left(\frac{0}{4}\right)\right]$$

$$=3\left[0+8\sin^{-1}\left(\sin\frac{\pi}{2}\right)\right]=3\left[8\frac{\pi}{2}\right]=12\pi \text{ sq. units.}$$

Exple 4 Find the area of the region bounded by $y=mx$, x axis and $x=2$.



Sol:- Reqd Area $=OABO$

$$=\int_0^2 y\,dx=\int_0^2 mx\,dx$$

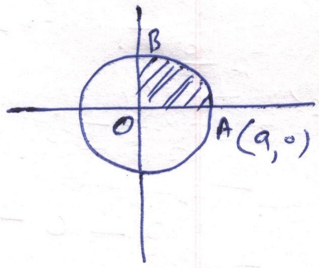
$$=m\left[\frac{x^2}{2}\right]_0^2=\frac{m}{2}\left[2^2-0\right]=\frac{m}{2}(4)=2m \text{ sq. units.}$$

Example 5 Find the whole area of the circle $x^2+y^2=a^2$

Sol :- The given circle $x^2+y^2=a^2$, whose centre is $(0,0)$ and radius is a

Now $x^2+y^2=a^2 \Rightarrow y^2=a^2-x^2 \Rightarrow y=\sqrt{a^2-x^2}$

Required Area $= 4\times(\text{Area of } OAB)$



$$= 4\int_0^a y\,dx$$

$$= 4\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4\left[\frac{a}{2}\sqrt{a^2-a^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)-\left(0+\frac{a^2}{2}\sin^{-1}(0)\right)\right]$$

$$= 4\left[0+\frac{a^2}{2}\sin^{-1}\left(\sin\frac{\pi}{2}\right)\right] = \frac{4}{2}a^2\frac{\pi}{2} = \pi a^2 \text{ sq. units.}$$

Volume of a solid formed by Revolution an area about axis

Case I :- If rotated about x-axis then

Total volume $V=\int_a^b \pi y^2\,dx$

Case II :- If rotated about y-axis then

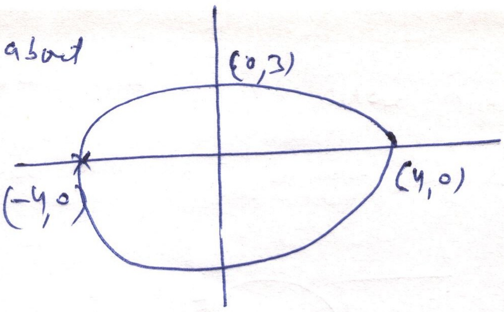
Total volume $V=\int_c^d \pi x^2\,dy$

Example 1 Find the volume generated by revolving the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ about the x axis

Sol :- Given eq $\frac{x^2}{16}+\frac{y^2}{9}=1 \Rightarrow \frac{y^2}{9}=1-\frac{x^2}{16}$

$$\Rightarrow y=\frac{3}{4}\sqrt{4^2-x^2}$$

Here we have revolve about x-axis between $x = -4$ and $x = 4$



Hence

$$V = \int_a^b \pi y^2 \, dx$$

$$= \pi \int_{-4}^{4} \frac{3}{4} (4^2 - x^2) \, dx = \frac{3\pi}{4} \left[16x - \frac{x^3}{3} \right]_{-4}^{4}$$

$$= \frac{3\pi}{4} \left[\left(16(4) - \frac{4^3}{3} \right) - \left(16(-4) - \frac{(-4)^3}{3} \right) \right]$$

$$= \frac{3\pi}{4} \left[\left(64 - \frac{64}{3} \right) - \left(-64 + \frac{64}{3} \right) \right]$$

$$= \frac{3\pi}{4} \left[128 - \frac{128}{3} \right]$$

$$= \frac{3\pi}{4} \left[128 \left(\frac{2}{3} \right) \right] = 64\pi \text{ cubic units}$$

Example 2:- Find the volume of the solid generated by the revolution of the curve $y^2 = 25x$ between the $x = 1$ and $x = 2$ about x-axis

Sol:- The given curve $y^2 = 25x$, $x = 1$, $x = 2$

Required volume $V = \int_a^b \pi y^2 \, dx = \pi \int_1^2 25x \, dx = 25 \int_1^2 \pi x \, dx$

$$= 25\pi \left[\frac{x^2}{2} \right]_1^2 = 25\pi \left(\frac{4}{2} - \frac{1}{2} \right)$$

$$= 25 \left(\frac{3}{2} \right) \pi = \frac{75\pi}{2} \text{ cubic units}$$

Average or Mean Value of a function

The average or mean value of a function $f(x)$ between $x = a$ and $x = b$ is defined as

$$\text{Average value} = \frac{\int_a^b f(x) \, dx}{b - a}$$

Example ① Find the Average value of the function $\cos^2 wt$ from $t=0$ to $t=\frac{2\pi}{w}$

Sol :- Here $f(t) = \cos^2 wt$, $t=0$ and $t=\frac{2\pi}{w}$

$$A\cdot V = \int_a^b \frac{f(t)\,dt}{b-a} = \frac{\int_0^{\frac{2\pi}{w}} \cos^2 wt\,dt}{\frac{2\pi}{w}-0}$$

$$= \frac{\int_0^{2\pi/w} (1+\cos 2wt)\,dt}{\frac{2\pi}{w}\times 2} \quad \left[\because 1+\cos 2\theta = 2\cos^2\theta\right]$$

$$= \frac{w}{4\pi}\int_0^{2\pi/w} (1+\cos 2wt)\,dt$$

$$= \frac{w}{4\pi}\left[t + \frac{\sin 2wt}{2w}\right]_0^{2\pi/w}$$

$$= \frac{w}{4\pi}\left[\frac{2\pi}{w} + \frac{\sin 2\left(\frac{2\pi}{w}\right)w}{2w}\right]$$

$$= \frac{w}{4\pi}\left[\frac{2\pi}{w} + 0\right] = \frac{1}{2}$$

Example ② Find the A.V of $f(x) = 2\cos x - \sin 2x$, from $x=0$ to $\frac{\pi}{2}$

Sol :

$$A\cdot V = \frac{\int_a^b f(x)\,dx}{b-a} = \frac{\int_0^{\pi/2} (2\cos x - \sin 2x)\,dx}{\pi/2 - 0}$$

$$= \frac{2}{\pi}\int_0^{\pi/2} (2\cos x - \sin 2x)\,dx = \frac{2}{\pi}\left[2\sin x + \frac{\cos 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{2}{\pi}\left[\left(2\sin\frac{\pi}{2} + \frac{\cos\pi}{2}\right) - \left(2\sin 0 + \frac{\cos 0}{2}\right)\right]$$

$$= \frac{2}{\pi}\left[2(1) + \frac{(-1)}{2} - 0 - \frac{1}{2}\right]$$

$$= \frac{2}{\pi}\left[2-1\right] = \frac{2}{\pi}$$

Root mean square value (R.M.S.V)

Root mean square value of a function $f(x)$ between $x=a$ and $x=b$ is defined by

$$\text{Root Mean Square Value} = \sqrt{\frac{\int_a^b [f(x)]^2\,dx}{b-a}}$$

Example (1) Find the Root mean square value of $f(t) = 2\sin 3t$ from $t=0$ to $\frac{\pi}{3}$

Sol:

$$\text{Root mean square value} = \sqrt{\frac{\int_a^b (f(t))^2\,dt}{b-a}}$$

$$= \sqrt{\frac{\int_0^{\pi/3} [2\sin 3t]^2\,dt}{\frac{\pi}{3}-0}} = \sqrt{\frac{3}{\pi}\times 4\int_0^{\pi/3} \sin^2 3t\,dt}$$

$$= \sqrt{\frac{12}{\pi}\int_0^{\pi/3} \frac{1-\cos 6t}{2}\,dt} \qquad \left[\because 1-\cos 2\theta = 2\sin^2\theta\right]$$

$$= \sqrt{\frac{12}{\pi}\times\frac{1}{2}\left[t-\frac{\sin 6t}{6}\right]_0^{\pi/3}}$$

$$= \sqrt{\frac{6}{\pi}\left(\frac{\pi}{3}-\frac{\sin 6(\pi/3)}{6}\right]-0}$$

$$= \sqrt{\frac{6}{\pi}\left(\frac{\pi}{3}-0\right)-0}$$

$$= \sqrt{\frac{6\pi}{3\pi}} = \sqrt{2}.$$

Exercise

Q(1) Find the area of region bounded by $y^2 = 4x$, $x=1$, $x=4$ and the x axis in Ist quadrant.

Q(2) Find the area of region bounded by $x^2 = y-3$, $y=4$, $y=6$ and y axis in Ist quadrant.

Q(3) Prove that the area of the ~~region~~ ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab

Q(4) Find the area of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$

Q(5) Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the x axis, between $x=0$, and $x=a$

Q(6) Using Integration find the area of the circle $x^2+y^2=r^2$

Q(7) Find the area of the region bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

Q(8) Find the area under a curve $y=\cos x$, $0 \le x \le \pi/2$ between x axis and the lines $x=0$ and $x=\pi/2$

Q(9) Find the volume of a solid generates by the revolution of area between parabola $y^2=4ax$ and its Latus rectum about x axis

Q(10) Find the volume of the solid of revolution obtained by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the minor axis

Q(11) Find the Volume of the the solid of revolution obtained by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the major axis.

Q(12) Find the volume of the sphere of radius a.

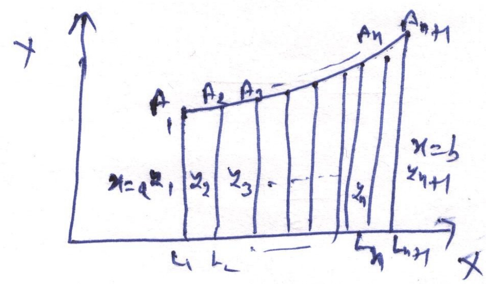
Q(13) Find the Average Value of $y=x^2+5x+2$, from $x=1$ to $x=4$

In Numerical Integration we well study two methods

(i) Trapezoidal Rule

(ii) Simpson's rule.

(i) Trapezoidal Rule :— Let $y = f(x)$ be a function having graph between $x = a$ and $x = b$ as shown in figure below:



Let us divide the interval $[a, b]$ into n-Subinterval. We drawn the $(n+1)$ ordinates i.e y_1, y_2 —— y_{n+1}.

Width of each subinterval $h = \frac{b-a}{n}$

∴ The Area bounded by curve $y = f(x)$, x axis and the ordinates $x = a$, and $x = b$ is

$$\int_a^b f(x)\,dx = \int_a^b y\,dx = \text{Sum of Areas of all Trapeziums.}$$

$$= \frac{h}{2}(y_1+y_2) + \frac{h}{2}(y_2+y_3) + \text{———} + \frac{h}{2}(y_n+y_{n+1})$$

$$= \frac{h}{2}\left[(y_1+y_{n+1}) + 2(y_2+y_3+y_5 \text{ ——— } + y_n)\right]$$

$$\boxed{\therefore \int_a^b y\,dx = \frac{h}{2}\left[(y_1+y_{n+1}) + 2(y_2+y_3+ \text{———} + y_n)\right]}$$

where $h = \frac{b-a}{n}$

Let $y = f(x)$ be a function defined on $[a,b]$. Let the interval $[a,b]$ be divided into $2n$ equal subintervals, then the ordinates is

y_1, y_2 ——— y_{2n}, y_{2n+1},

Now Required Area is

$$\boxed{\int_a^b y\,dx = \frac{h}{3}\left[(y_1 + y_{2n+1}) + 2(y_3 + y_5 + \text{---} \; y_{2n-1}) + 4(y_2 + y_4 + \text{---} + y_{2n})\right]}$$

where $h = \dfrac{b-a}{n}$

<u>Remark</u> (I) In Trapezodial Rule n may be even or odd.

(II) In Simpson's rule n is always even

(III) n is number of interval. i.e If $n+1$ is ordinates the $(n+1-1) = n$ is interval..

<u>Example</u> ① Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ by Simpson's rule by dividing the interval into four equal parts and also find approximate value of π

<u>Sol</u>:- Here interval $[0, 1]$ into four equal parts so

$$h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$$

Given $y = \dfrac{1}{1+x^2} \Rightarrow y_1, y_2, y_3$ ——— y_5 be five ordinates

when $x = 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}$

$$y_1 = \frac{1}{1+0^2} \quad , \quad y_2 = \frac{1}{1+\left(\frac{1}{4}\right)^2} = \frac{16}{17} = .9411$$

$$y_3 = \frac{1}{1+\left(\frac{1}{2}\right)^2} = \frac{4}{5} = .8 \quad , \quad y_4 = \frac{1}{1+\left(\frac{3}{4}\right)^2} = \frac{16}{25} = .64$$

$$y_5 = \frac{1}{1+1^2} = \frac{1}{2} = .5$$

By Simpson's Rule, we get

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[(y_1+y_5)+4(y_2+y_4)+2(y_3)\right]$$

$$= \frac{1}{12}\left[(1+.5)+2(.8)+4(.9411+.64)\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{1}{12}\left[1.5+1.6+6.3244\right] = .7854 \quad \text{——— (1)}$$

Now $\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}\right)\right] - 0$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4} \quad \text{——— (2)}$$

From (1) and (2), $\frac{\pi}{4} = .7854 \Rightarrow \pi = 3.1416.$

Example 2: - Use Trapezoidal Rule to evaluate $\int_4^8 \frac{dx}{x-3}$ by taking 5 ordinates.

Sol: - Let $y = \frac{1}{x-3}$, Here ordinate is 5 so number of interval is $= 4$, then $h = \frac{b-a}{n} = \frac{8-4}{4} = \frac{4}{4} = 1$

Let y_1, y_2, y_3, y_4, y_5 be the ordinate then $x = 4, 5, 6, 7, 8$

Now $y_1 = \frac{1}{4-3} = 1$, $y_2 = \frac{1}{5-3} = \frac{1}{2} = .5$

$6-3 \qquad 3$

$$y_4 = \frac{1}{7-3} = \frac{1}{4} = .25$$

$$y_5 = \frac{1}{8-3} = \frac{1}{5} = .20$$

By Trapezoidal Rule, we get

$$\int_4^8 \frac{dx}{x-3} = \frac{h}{2}\left[(y_1+y_5)+2(y_2+y_3+y_4)\right]$$

$$= \frac{1}{2}\left[(1+.2)+2(.5+.33+.25)\right]$$

$$= \frac{1}{2}\left[1.2+2(1.083)\right] = \frac{1}{2}\left[3.366\right] = 1.683$$

Exple ③ Find the approximate area by Trapezoidal rule under the curve whose ordinates are given as below

$x =$	0	1	2	3	4	5
$y =$	0	2.5	3	4.5	5	7.5

Sol: Here $a=0$, $b=5$, $n=5$

$$\therefore h = \frac{b-a}{n} = \frac{5-0}{5} = 1$$

By Trapezoidal Rule.

$$\text{Area} = \frac{h}{2}\left[(y_1+y_5)+2(y_2+y_3+y_4)\right]$$

$$= \frac{1}{2}\left[(0+7.5)+2(2.5+3+4.5+5)\right]$$

$$= \frac{1}{2}\left[7.5+30\right] = \frac{37.5}{2}$$

$$= 18.75$$

$e^3 = 20.09$, $e^4 = 54.60$, find the value of $\int_0^4 e^x dx$ by Simpson's rule. find al~~so~~so exact value.

Sol:- Here $\int_0^4 e^x dx$, $y = e^x$

$$h = \frac{4-0}{4} = 1$$

Here $n = 4$ because (ordinate is 5)

The given table

$x =$	0	1	2	3	4
$y =$	1	2.72	7.39	20.09	54.60
	y_1	y_2	y_3	y_4	y_5

∴ By Simpson's rule.

$$\int_0^4 e^x dx = \frac{h}{3}\left[(y_1 + y_5) + 2(y_3) + 4(y_2 + y_4)\right]$$

$$= \frac{1}{3}\left[(1 + 54.60) + 2(7.39) + 4(2.72 + 20.09)\right]$$

$$= \frac{1}{3}\left[55.60 + 18.78 + 91.24\right]$$

$$= \frac{161.62}{3} = 53.87.$$

Exact value $= \int_0^4 e^x dx = \left[e^x\right]_0^4 = e^4 - e^0$

$$= 54.60 - 1 = 53.60.$$

Q(1): Apply Simpson's rule of app. integration to find the value of $\int_4^8 \frac{dx}{x}$ using 4 equal intervals.

Ans (0.694)

Q(2) Evaluate approximate the value of $\int_{-3}^{3} x^4 dx$ by Simpson's rule by taking seven ordinates

Ans (97.2)

Q(3) Calculate the approximate values of the interval $\int_1^5 (126-x^3)^{1/2} dx$ using Simpson's rule by taking five ordinates.

Ans (35.63)

Q(4) Evaluate $\int_0^{2.5} \sqrt{16-x^2}\, dx$ using Trapezoidal rule by taking six ordinates.

Ans (9.282)

Q(5) Using Simpsons' rule to evaluate $\int_1^2 \frac{1}{x} dx$ by taking four equal parts. Hence find the value of $\log 2$.

Ans (0.693)

(6) Evaluate by Simpson's rule the value of $\int_0^4 (1+2x^3)^{1/2} dx$ using five ordinates

(Ans) (19.0653)

(7) Evaluate $\int_0^1 \frac{1}{1+x} dx$, by Trapezoidal rule taking eight equal intervals.

Ans (.694)

(8) Applying Trapezoidal Rule to evaluate $\int_0^8 \frac{1}{x+3} dx$ by taking four equal intervals.

(9) The velocity of a body moving in a straight line at given below

t =	0	1	2	3	4	5
V =	4	3.5	3	2.5	2	1.5

Ans (13.75)

The using Trapezoidal rule find the distance ~~travelled~~ in 5 sec.