(II) Maxima and Minima

Maxima and Minima are local properties. A function $f(x)$ has a maximum at x_0 if $f(x_0)$ is greatest in some interval (x_0-h, x_0+h) or simply to say, in some interval containing x_0 and likewise minima exists at x_0 if in some interval containing x_0, $f(x_0)$ is least.

Thus $f(x)$ has maximum at x_0 if in some interval

$$I =]x_0-h, x_0+h[, \text{ for all } x \in I, f(x) < f(x_0)$$

and Minima if $f(x) > f(x_0)$

Note: $f(x)$ has maximum at x_0 does not mean that $f(x_0)$ is greatest value of $f(x)$ in the domain D of $f(x)$ and likewise minimum at x_0 does not mean that $f(x_0)$ is the least value of $f(x)$ in the domain D.

Necessary condition for maxima and Minima:

Let $f(x)$ be defined in the interval D. Then maxima or minima may exist at pt $x_0 \in D$ if

(I) $f'(x_0) = 0$

(II) $f'(x_0)$ does not exist (i.e $f(x)$ is non differentiable at x_0) but is continuous at x_0

The general term used for maxima and minima is known as extremum values.

$$\frac{1-15h}{h-100h^2} \simeq \frac{1}{h} > 0$$

<u>Working Method</u>

Thus to check the maxima or minima at $x=a$, first find the sign of $f'(x)$ for values of x slightly less that a (i.e $x_0 < a$) and then find the sign of $f'(x)$ for values of x slightly greater that a (i.e $x_0 > a$)

(i) If the sign of $f'(x)$ changes from +ve to −ve, then $f(x)$ has a maximum at $x=a$

(ii) If the sign of $f'(x)$ changes from −ve to +ve, then $f(x)$ has a minimum at $x=a$.

(iii) If the sign of $f'(x)$ does not change, then $f(x)$ has neither maximum nor minimum at $x=a$.

(II) <u>Second Derivative Test</u>: If $f(x)$ has a derivative at x_0 and $f'(x_0)=0$. Then

(i) $f''(x_0) < 0 \Rightarrow f(x)$ has maxima at x_0

(ii) $f''(x_0) > 0 \Rightarrow f(x)$ has minima at x_0

(iii) If $f''(x_0) = 0$, we cannot say anything

(iv) If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then $f(x)$ has

Example ① Find the pts of maxima and minima and the corresponding maximum and minimum values of each of following if any

(i) The constant function α (ii) $\frac{1}{x^2+2}$

(iii) $\sin x - \cos x$, $0 < x < 2\pi$

Solution: (i) Let $f(x) = \alpha$

$\Rightarrow f'(x) = 0$ for all values of x

$\Rightarrow f''(x) = 0$

$\therefore$ The second derivative test fails.

(ii)

Let $f(x) = \frac{1}{x^2+2} = (x^2+2)^{-1}$

$\Rightarrow f'(x) = \frac{-1}{(x^2+2)^2} \frac{d}{dx}(x^2+2)$

$\Rightarrow f'(x) = \frac{-2x}{(x^2+2)^2}$

For maxima or minima $f'(x) = 0 \Rightarrow \frac{-2x}{(x^2+2)^2} = 0$

$\Rightarrow \boxed{x = 0}$

Now $f''(x) = \frac{(x^2+2)^2 \frac{d}{dx}(-2x) - (-2x)\frac{d}{dx}(x^2+2)^2}{(x^2+2)^4}$

$\Rightarrow f''(x) = \frac{-8}{4}$ at $x = 0$

$\Rightarrow f''(x) = -2 \Rightarrow f''(x) < 0$

$\Rightarrow f(x)$ has maxima at $x = 0$

at $x = \frac{7\pi}{4}$, $f''(x) = \cos x - \sin x$

$$\therefore \quad f''\left(\frac{7\pi}{4}\right) = \cos\frac{7\pi}{4} - \sin\frac{7\pi}{4}$$

$$= \cos\left(2\pi - \frac{\pi}{4}\right) - \sin\left(2\pi - \frac{\pi}{4}\right)$$

$$= \cos\pi/4 + \sin\pi/4 \quad [\text{IV}^{th} \text{ quadrant}]$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

$\therefore f''(x) > 0$

$\Rightarrow f(x)$ a minimum at $x = \frac{7\pi}{4}$

and value $f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4}$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$$

<u>Example ②</u> Prove that the following functions do not have maxima or Minima:

(a) $f(x) = x+2$ (b) e^x (c) $\log x$

<u>Sol</u>. (a) We have, $f(x) = x+2$

$\Rightarrow f'(x) = 1$

$\Rightarrow f''(x) = 0$

For maxima or minima, $f'(x) = 0$

$\Rightarrow 1 = 0$, which is not possible

Hence $f(x) = x+2$ does not have maxima or Minima

$$4x^3 - 124x + a = 0$$

$$\Rightarrow 4(1)^2 - 124 \times 1 + a = 0 \qquad [\because x = 1]$$

$$\Rightarrow 4 - 124 + a = 0$$

$$\Rightarrow \boxed{a = 120}$$

and $f''(x) = 12x - 124$

$$\Rightarrow f''(1) = 12 \times 1 - 124 = -112$$

$$\Rightarrow f''(x) < 0 \quad \text{at } x = 1$$

$\Rightarrow f(x)$ has a maximum at $x = 1$, when $a = 120$.

Hence $a = 120$.

Example (4) Prove that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{\frac{1}{e}}$

Sol: Let $y = \left(\frac{1}{x}\right)^x \Rightarrow \log y = x \log\left(\frac{1}{x}\right)$

$$\Rightarrow \log y = x[\log 1 - \log x]$$

$$\Rightarrow \log y = x[-\log x]$$

$$\Rightarrow \log y = -x \log x$$

Diff of both sides w.r.to x, we have

$$\frac{1}{y}\frac{dy}{dx} = -\left[x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x)\right]$$

$$= -\left[x \times \frac{1}{x} + x \log x\right]$$

$$\therefore \frac{dy}{dx} = -y(1 + x\log x).$$

$$\therefore \frac{dy}{dx} = -\left(\frac{1}{x}\right)^x [1 + x \log x] \longrightarrow (1)$$

Example (5)

Prove that the function $f(x) = x^3 - 3x^2 + 3x + 7$ at $x = 1$ neither maximum nor minimum.

Sol: Here given, $f(x) = x^3 - 3x^2 + 3x + 7$

$\Rightarrow f'(x) = 3x^2 - 6x + 3$

$\Rightarrow f''(x) = 6x - 6$

For max or minima, $f'(x) = 0$

$\Rightarrow 3x^2 - 6x + 3 = 0 \Rightarrow x^2 - 2x + 1 = 0$

$\Rightarrow (x-1)^2 = 0$

$\Rightarrow x = 1$

At $x = 1$, $f''(1) = 6 - 6 = 0$

when $f''(x) = 0$ then $f'''(x) = 6 \neq 0$

Hence $f(x)$ neither maximum nor minimum at $x = 1$.

Example 6 Find the maximum profit that a company can make, if the profit function is given by $p(x) = 41 - 24x - 18x^2$.

Sol: Profit function in $p(x) = 41 - 24x - 18x^2$

$\Rightarrow p'(x) = -24 - 36x$

$\Rightarrow p''(x) = -36$

For maxima or minima, $p'(x) = 0$

$\Rightarrow -24 - 36x = 0$

$\Rightarrow 2 + 3x = 0$

Exercise (I)

(1) Find the pts of maxima and minima, if any of the following functions. Find also the maximum and minimum values

(i) $f(x) = x^3 - 12x$ (ii) $f(x) = \sin 2x$, $0 < x < \pi$

(iii) $f(x) = \sin x - \cos x$, $0 < x < 2\pi$ (iv) $f(x) \frac{x}{2} + \frac{2}{x}$, $x > 0$

(2) Find the maximum value of $x e^{x}$.

(3) Prove that $\sin x + \cos x$ has maximum value $\sqrt{2}$.

(4) Prove that the ~~maximum~~ minimum value of x^x is $\left(\frac{1}{e}\right)^{\frac{1}{e}}$

(5) Show that $\frac{\log x}{x}$ has a maximum value at $x = e$.

(6) Prove that the maximum value of $(x)^{1/x}$ is $(e)^{1/e}$

(7) Find the pts of maximum or minimum and the corresponding maximum and minimum values of the functions
$$f(x) = \sin 2x - x \quad , -\frac{\pi}{2} < x < \pi/2$$

(8) Find all the pts of maximum and minimum value of the function: $f(x) = 2x^3 - 21x^2 + 36x - 20$.

Application of Maxima or Minima

Example (1) If $x+y=10$, then find the maximum value of xy.

Sol'
$$x+y=10$$
$$\Rightarrow y = 10-x \quad \text{——— (1)}$$

Now let $f(x) = xy$
$$\Rightarrow f(x) = x(10-x) \quad [\text{from (1)}]$$
$$\Rightarrow f(x) = 10x - x^2$$

$$\therefore f'(x) = 10-2x$$
$$\Rightarrow f''(x) = -2$$

For maxima or minima, $f'(x)=0$
$$\Rightarrow 10-2x=0 \Rightarrow \boxed{x=5}$$

So $f''(x) < 0$ for all value of x

$\therefore$ $f(x)$ is maximum at $x=5$

Now maximum value of $f(x) = xy$
$$= 5\times 5 \qquad \left[\begin{array}{l} y = 10-x \\ \Rightarrow y=5 \end{array}\right]$$
$$= 25.$$

Ans I

1 (i) Min at $x = 2$ is -16, Max at $x = -2$ is 16

(ii) Max at $x = \frac{\pi}{4}$ is 1, Min at $x = \frac{3\pi}{4}$ is -1

(iii) Max at $x = \frac{2\pi}{4}$ is $\sqrt{2}$, Min at $\frac{7\pi}{4}$ is $-\sqrt{2}$

(iv) Max at $x = 2$, is 2

2 $\frac{1}{e}$ 7 Max at $x = \pi/6$ is $\frac{\sqrt{3}}{2} - \pi/6$, Min at $x = -\pi/6$ is $-\frac{\sqrt{3}}{2} + \frac{\pi}{6}$

8 Max at $x = 1$ is -3, Min at $x = 6$ is -128.

Example ② Find two positive numbers whose sum is 14 and the sum of whose squares is minimum.

Sol: Let one part is x

the other part is $14-x$

According to question, $x^2+(14-x)^2 = S$

$$\Rightarrow \frac{dS}{dx} = 2x + 2(14-x)\cdot(-1)$$

$$\Rightarrow \frac{dS}{dx} = 2x - 28 + 2x$$

$$\frac{dS}{dx} = 4x - 28$$

$$\Rightarrow \frac{d^2S}{dx^2} = 4$$

For max or minima, $\frac{dS}{dx} = 0 \Rightarrow 4x - 28 = 0$

$$\Rightarrow 4x = 28 \Rightarrow x = 7$$

Now $\frac{d^2S}{dx^2} > 0$ for $x = 7$.

Here S is minimum when the number are 7, 14-7, i.e 7 and 7.

Example (2) ~~First~~ If the parameter of a rectangle is 100cm then find the sides of rectangle is area is maximum.

Sol:. Let x and y be the sides of rectangle then

Parameter of rectangle $= 2(x+y)$

$\Rightarrow 100 = 2(x+y)$

$\Rightarrow x+y = 50$ ——— (1)

Now Area, $A = xy$

$\Rightarrow A = x(50-x)$ (from (1))

$\Rightarrow A = 50x - x^2$

$\Rightarrow \frac{dA}{dx} = 50 - 2x$

$\Rightarrow \frac{d^2A}{dx^2} = -2$

For maximum or Minimum, $\frac{dA}{dx} = 0 \Rightarrow 50 - 2x = 0$

$\Rightarrow x = 25$ cm

Now $\frac{d^2A}{dx^2} < 0$

Here Area is maximum at $x = 25$:

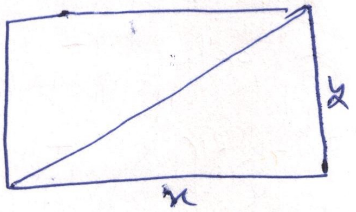
Now $y = 50 - x = 50 - 25 = 25$

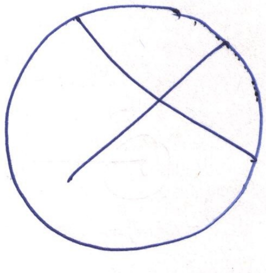
Hence sides of rectangle is 25cm and 25cm.

Example (7)

Show that, of all rectangles with a given parameter, the square has the largest area.

Sol: Let x and y be the sides of the Rectangle.





Here Parameter is given

$$\therefore \; P = 2(x+y) \Rightarrow \frac{P-2x}{2} = y \quad \text{——} (1)$$

Now Area, $A = xy = x\left(\frac{P-2x}{2}\right)$ (From (1))

$$A = \frac{xP - 2x^2}{2}$$

$$\therefore \frac{dA}{dx} = \frac{P-4x}{2}$$

$$\Rightarrow \frac{d^2A}{dx^2} = \frac{-4}{2} = -2$$

For maxima or Minima, $\frac{dA}{dx} = 0 \Rightarrow \frac{P-4x}{2} = 0$

$$\Rightarrow P - 4x = 0$$

$$\Rightarrow P = 4x$$

$$\therefore y = \frac{4x-2x}{2} = \frac{2x}{2} = x \quad [\text{From } (1)]$$

$$\Rightarrow \boxed{y = x}$$

Hence area is max, when $x = y$, ie when rectangle becomes a square.

Example (5) A wire of lenght 28 m is to be cut into two pieces. one of the peices is to be made into a square and the other into a circle. what should be the lengths of the two pieces so that the combined area of the square and the circle is minimum.

<u>Sol:</u> Let one part be of lenght x, then the other part $= 28 - x$.

Let the part of length x be converted into a circle of radius r.

$$\therefore \quad x = 2\pi r \text{ (parameter of circle)}$$

$$\Rightarrow r = \frac{x}{2\pi} \quad \text{———} \quad (1)$$

Now Area of circle, $A_1 = \pi r^2$

$$A_1 = \pi \left(\frac{x}{2\pi}\right)^2 \quad [\text{from } (1)]$$

$$A_1 = \frac{\pi x^2}{4\pi^2} = \frac{x^2}{4\pi}$$

$$\therefore A_1 = \frac{x^2}{4\pi} \quad \text{———} \quad (2)$$

Now the second part of length $28 - x$ is converted into of a square of side $\frac{1}{4}(28-x)$

Area of Square = (Sides)^2

i.e $A_2 = \left[\frac{1}{4}(28-x)\right]^2$

$A_2 = \frac{1}{16}(28-x)^2$ ——— (3)

Now total Area, $A = A_1 + A_2$

$\Rightarrow A = \frac{x^2}{4\pi} + \frac{1}{16}(28-x)^2$ [From (2) and (3)]

$\therefore \frac{dA}{dx} = \frac{2x}{4\pi} + \frac{2}{16}(28-x)(-1)$

$= \frac{2x}{4\pi} - \frac{1}{8}(28-x)$

$\Rightarrow \frac{d^2A}{dx^2} = \frac{2}{4\pi} + \frac{1}{8} = +ve$

$\Rightarrow \frac{d^2A}{dx^2} > 0$

$\Rightarrow$ Area is minimum.

For Maximum or Minimum, $\frac{dA}{dx} = 0$

$\Rightarrow \frac{2x}{4\pi} - \frac{1}{8}(28-x) = 0$

$\Rightarrow \frac{x}{2\pi} = \frac{28-x}{8}$

$\Rightarrow 4x = 28\pi - x\pi$

$\Rightarrow 4x + x\pi = 28\pi$

$$\Rightarrow x(4+\pi) = 28\pi$$

$$\Rightarrow x = \frac{28\pi}{4+\pi}$$

One part $= \frac{28\pi}{4+\pi}$

and other part $= 28 - \left(\frac{28\pi}{4+\pi}\right)$

$$= \frac{112 + 28\pi - 28\pi}{4+\pi}$$

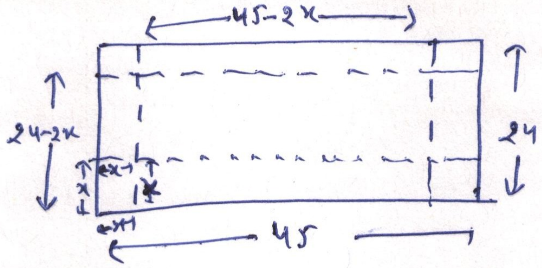
$$= \frac{112}{4+\pi}.$$

Example 6 :- A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so the volume of the box is maximum possible?

Sol :- Let each side of the square cutt off from each corner be x cm.

$\therefore$ Sides of the rectangular box are $(45-2x)$ cm and $(24-2x)$ cm.

Volume of the box $= l \times b \times h$



$$\therefore\ V = (45-2x)(24-2x)(x)$$

$$V = 2(2x^3 - 69x^2 + 540x)$$

$$\therefore \frac{dV}{dx} = 2(6x^2 - 138x + 540)$$

$$\Rightarrow \frac{dV}{dx} = 12(x^2 - 23x + 90)$$

$$\Rightarrow \frac{d^2V}{dx^2} = 12(2x - 23)$$

For Maxima or Minima,

$$\frac{dV}{dx} = 0$$

$$\Rightarrow 12(x^2 - 23x + 90) = 0$$

$$\Rightarrow x^2 - 23x + 90 = 0$$

$$\Rightarrow (x-5)(x-18) = 0$$

$$\Rightarrow x = 5,\ 18$$

For, $x = 5$, $\frac{d^2V}{dx^2} = 12(10-23) = -13 \times 12$

$$= -156$$

$$\therefore \frac{d^2V}{dx^2} < 0$$

$\Rightarrow$ Volume is maxim at $x = 5$

~~for x = 18~~

But x cannot be grater than 12.

So $x = 18$ not possible.

Hence, At $x = 5$ volume is maximum

So the sides of rectangular is

$(45 - 2x)$ cm and $(24 - 2x)$ cm

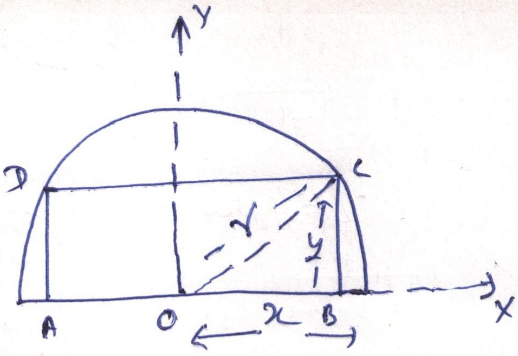
ie $(45 - 10)$ cm and $(24 - 10)$ cm

$\Rightarrow$ 35 cm and 14 cm.

Example 7 A rectangle is inscribed in a semi-circle of radius r with one of its sides on the diameter of semi-circle. Find the dimensions of the rectangle so that its area is maximum, find also the area.

Sol: From the figure

$$x^2 + y^2 = r^2 \quad \text{--- (1)}$$



where $OB = x$ and $BC = y$

radius of circle $OC = r$ (given)

Length of rectangle $AB = 2x$

and breath of rectangle $BC = y$

From (1), we have

$$y^2 = r^2 - x^2 \Rightarrow y = \sqrt{r^2 - x^2} \quad \text{—— (2)}$$

$\therefore$ Area of rectangle $= l \times b = 2x \times y$

$$\therefore \quad A = 2x\sqrt{(r^2 - x^2)} \qquad [\text{From (2)}]$$

$$\Rightarrow A^2 = 4x^2 (r^2 - x^2)$$

$$\Rightarrow \underset{(\text{let})}{u} = 4r^2x^2 - 4x^4$$

$$\Rightarrow \frac{du}{dx} = 8r^2x - 16x^3$$

$$\Rightarrow \frac{d^2u}{dx^2} = 8r^2 - 48x^2$$

For Maxima or Minima

$$\frac{dy}{dx} = 0$$

$$\Rightarrow 8r^2x - 16x^3 = 0$$

$$\Rightarrow 8x(r^2 - 2x^2) = 0$$

$$\Rightarrow r^2 - 2x^2 = 0$$

$$\Rightarrow 2x^2 = r^2$$

$$\Rightarrow x = \frac{r}{\sqrt{2}}$$

when $x = \frac{r}{\sqrt{2}}$ then

$$\frac{d^2y}{dx^2} = 8r^2 - 24r^2 = -16r^2$$
$$= -ve$$

$$\Rightarrow \frac{d^2y}{dx^2} < 0$$

$\Rightarrow$ Area is maximum when $x = \frac{r}{\sqrt{2}}$

So the side of rectangle are $2x$ and $\sqrt{r^2 - x^2}$

i.e. $2 \times \frac{r}{\sqrt{2}}$ and $\sqrt{r^2 - \frac{r^2}{2}}$

i.e $\sqrt{2}r$ and $\frac{r}{\sqrt{2}}$.

Now, Area, $A = 2x \times y = 2 \times \frac{r}{\sqrt{2}} \times \frac{r}{\sqrt{2}} = r^2$.

Example 8. Show that the right circular cylinder of given surface area maximum volume is such that its hight is equal to the diamenter of the base.

Sol:- Let S be the given surface area of the cylinder whose radius is r and hight h. then

$$S = 2\pi r h + 2\pi r^2$$

$$\Rightarrow 2\pi r h = S - 2\pi r^2$$

$$\Rightarrow h = \frac{S - 2\pi r^2}{2\pi r} \quad \text{——— (1)}$$

Volume, $V = \pi r^2 h$

$$\Rightarrow V = \pi r^2 \left[\frac{S - 2\pi r^2}{2\pi r} \right] \quad [\text{From (1)}]$$

$$V = \frac{1}{2}\left[S r - 2\pi r^3 \right]$$

$$\therefore \frac{dV}{dr} = \frac{1}{2}\left[S - 6\pi r^2 \right]$$

$$\therefore \frac{d^2V}{dr^2} = \frac{1}{2}\left[-12\pi r \right] = -6\pi r < 0$$

$\Rightarrow$ Volume is maximum

for maxima or minima,

$$\frac{dV}{dr} = 0 \implies \frac{1}{2}(S - 6\pi r^2) = 0$$

$$\implies S - 6\pi r^2 = 0$$

$$\implies S = 6\pi r^2 \quad \text{--- (2)}$$

But $S = 2\pi r^2 + 2\pi r h$

So $6\pi r^2 = 2\pi r^2 + 2\pi r h$

$$\implies 4\pi r^2 = 2\pi r h$$

$$\implies h = 2r \implies \boxed{r = \frac{h}{2}}$$

$\therefore$ Volume is maximum at $r = \frac{h}{2}$ or $h = 2r$

Hence volume is maximum when height of the cylinder is equal to the diameter of the base.

Exercise II

(1) Find two numbers whose sum is 24 and whose product is as large as possible

(2) Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

(3) Find two positive numbers x and y such that $x+y=60$ and xy^3 is maximum.

(4) Show that, off all rectangles inscribed in a given fixed circle, the square has the maximum area

(5) A wire of length 25m is to be cut into two pieces. One of the pieces is to be made into square and the other into a circle. what could be the lengths of the two pieces so that the combined area of the square and the circle is minimum.

(6) Find the largest possible area of a rectangle having parameter of 200 meters.

(7) Show that a closed right circular cylinder of total given surface and maximum volume is such that its hight is equal to the diameter of the base.

(8) Of all the closed cylinderical can, which enclose a given volume of 100 cubic cm. Find the dimensions of the can which has the min. surface area.

(9) A window is in the form of a rectangle surmounted by a semi-circular opening. If the perimeter of the window is 20 m. Find the dimensions of the window so as to admit max. possible light through the whole opening.

(10) A square piece of tin of side 18 cm is to be made into a box without top by cutting a square form each corner and folding up the flaps to form a box. What should be the side of the square to be cutt off so that the volume of the box is maximum.

(III) Derivative as a rate measure (1)

Let $y = f(x)$ be a relation between the variables x and y. If δx be a small change in x and δy be the small change in y, then $\frac{\delta y}{\delta x}$ represents the average rate of change in y w.r.to x in interval $(x, x+\delta x)$

Taking limit $\delta x \to 0$, the average rate of change $\frac{\delta y}{\delta x}$ becomes $\frac{dy}{dx}$, called the instantaneous rate of change in y w.r.to. x.

Note (i) If A be the area then rate of change of area w.r.to x is $\frac{dA}{dx}$.

(ii) If r be the radius then rate of change is $\frac{dr}{dt}$.

Velocity and acceleration:

If s is the distance moved by a particle in time t, then the velocity v and acceleration a of the particle at any instant t are given by

$$v = \frac{ds}{dt} \text{ and } a = \frac{dv}{dt} = \frac{d^2 s}{dt^2} = v\frac{dv}{dx}$$

velocity and acceleration both are +ve in the direction of s increasing with time t.

<u>Some Case</u> (i) If the acceleration of a partical is -ve it is called retardation.

(ii) If acceleration is zero then the partical is said to be moving with uniform velocity.

<u>Example</u>:- (1) Find the rate of change of the area of a circle with respect to its radius r, when $r = 5$ cm.

<u>Sol</u>: Let A be the area of the circle of radius r.

then,
$$A = \pi r^2$$
$$\Rightarrow \frac{dA}{dr} = 2\pi r$$

when, $r = 5$, then $\frac{dA}{dr} = 2\pi(5) = 10\pi \text{ cm}^2/\text{cm}$

Hence, rate of change of the area of the circle w.r.t. r is 10π cm^2/cm.

Emple (2)

The radius of a circle is increasing uniformly at the rate of 3cm/sec. Find the rate at which the area of the circle is increasing when the radius is 10 cm.

Sol:- Let r be the radius of the circle.

$\therefore$ Rate of change of radius $= \frac{dr}{dt} = 3 \text{ cm/sec}$.

$$\text{Area of circle}(A) = \pi r^2$$

$$\therefore \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$= 2\pi r (3)$$

$$\frac{dA}{dt} = 6\pi r$$

when $r = 10$, then $\frac{dA}{dt} = 6\pi(10)$

$$\Rightarrow \frac{dA}{dt} = 6\pi \frac{cm^2}{sec}$$

Emple (3) The distance x K.m. which a train travels in time t hours is given by the relation $x = t^3 + 3t^2 + t - 6$. Find the velocity and accelartion of the train at the end of 3 hours.

The given relation is

$$x = t^3 + 3t^2 + t - 6$$

$$\Rightarrow \frac{dx}{dt} = 3t^2 + 6t + 1$$

$$\Rightarrow \frac{d^2x}{dt^2} = 6t + 6$$

Now, velocity $v = \frac{dx}{dt} = 3t^2 + 6t + 1$

at $t = 3$, $v = \frac{dx}{dt} = 3(3)^2 + 6(3) + 1$

$$v = 27 + 18 + 1$$

$$v = 46 \ \frac{km}{hr}$$

Now Accelation $a = \frac{d^2x}{dt^2} = 6t + 6$

At $t = 3$, $a = \frac{d^2x}{dt^2} = 6 \times 3 + 6$

$$a = 24 \frac{Km}{hr^2}.$$

Example (4) The length x of a rectangle is decreasing at the rate of 3 cm/min and the width y is increasing at the rate of 2 cm/min, when $x = 10$ and $y = 6$ cm, find the rate of change of (a) the perimeter and (b) the area of the rectangle.

Sol:- Since the length x is decreasing and the width y is increasing, we have

$$\frac{dx}{dt} = -3 \text{ cm/min}$$

and
$$\frac{dy}{dt} = 2 \text{ cm/min}$$

(a) The perimeter P of the rectangle is

$$P = 2(x+y)$$

$$\therefore \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$\frac{dP}{dt} = 2(-3+2) = -2$$

Hence the rate of change of perimeter $= -2$

(b) The area of a rectangle is

$$A = xy$$

$$\therefore \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$\frac{dA}{dt} = 10(2) + 6(-3)$$

$[x = 10cm \text{ and } y = 6cm]$

$$= 20 - 18 = 2\, cm^2/sec.$$

Hence, rate of change in area is $2\, cm^2/sec$.

Example 5 A balloon which always remains spherical is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of the balloon increasing when the radius is 15 cm.

Sol: Volume of the spherical balloon,

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dt} = \frac{4}{3}\pi\, 3r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \text{———} \; (1)$$

Now given that $\frac{dV}{dt} = 900$ ——— (2)

From (1) and (2), we have

$$900 = 4\pi r^2 \frac{dr}{dt}$$

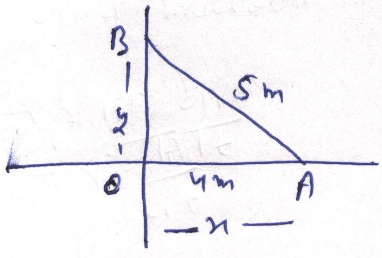
$$900 = 4 \times \frac{22}{7} \times (15)^2 \frac{dr}{dt}$$

$$\therefore \frac{dV}{dt} = \frac{900 \times 7}{4 \times 22 \times 15 \times 15} = \frac{7}{22}$$

Hence the radius of the balloon is increasing at the rate of $\frac{7}{22}$ cm/sec $= \frac{1}{11}$ cm/sec.

Example 6 A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 m/sec. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall.

Sol: Let AB be the ladder and OB be the wall.



Let $OA = x$ and $OB = y$

then $x^2 + y^2 = 5^2$

$$\Rightarrow x^2 + y^2 = 25 \quad \text{—— (1)}$$

Now $$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \quad \text{—— (2)}$$

when $x = 4$ then from (1) we have

$$4^2 + y^2 = 25 \Rightarrow y^2 = 25 - 16$$

$$y^2 = 9 \Rightarrow \boxed{y = 3}$$

Now, $\frac{dx}{dt} = 2$ (Given)

Putting the values in (2), we get

$$4 \times 2 + 3\frac{dy}{dt} = 0$$

$$\Rightarrow 3\frac{dy}{dt} = -8 \Rightarrow \frac{dy}{dt} = -8/3$$

Hence the hight of the ladder on the wall is decreasing at the rate 8/3 m/sec.

Example (7) A partical moves along the curve $6y = x^3 + 2$. Find the pts on the curve at which the y-coordinate 8 times as fas as the x-coordinate.

Sol: We have, $6y = x^3 + 2$ ——— (1)

$$\Rightarrow 6\frac{dy}{dt} = 3x^2\frac{dx}{dt}$$

$$\Rightarrow 2\frac{dy}{dt} = x^2\frac{dx}{dt} \quad \text{——— (2)}$$

But according to question, $\frac{dy}{dt} = 8\frac{dx}{dt}$ ——— (3)

Now, from (2) and (3), we get

$$2 \times 8 \frac{dx}{dt} = x^2 \frac{dx}{dt}$$

$$\Rightarrow x^2 = 16$$

$$\Rightarrow x = \pm 4$$

when $x = 4$, then from (1),

$$6y = 4^3 + 2$$

$$\Rightarrow y = \frac{64+2}{6} = \frac{66}{6} = 11$$

when $x = -4$, then from (1),

$$6y = (-4)^3 + 2 = -64 + 2$$

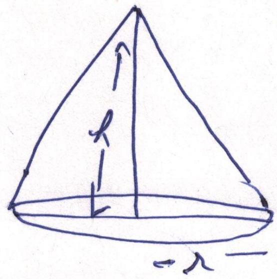
$$\Rightarrow y = -\frac{62}{6} = \frac{-31}{3}$$

Hence, the pts are. $(4, 11)$ and $\left(-4, \frac{-31}{3}\right)$

Example 8 Sand is pouring from a pipe at the rate of 12 cm^3/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand-cone increasing when the height is 4 cm.

Sol:-

Let r and h be the radius and height of the sand-cone at time t respectively.



According to question, we have

$$h = \frac{r}{6}$$

$$\Rightarrow \boxed{r = 6h}$$

The volume of cone, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (6h)^2 h$

$$\Rightarrow V = 12\pi h^3$$

$$\Rightarrow \frac{dV}{dt} = 36\pi h^2 \frac{dh}{dt} \quad \text{——— (1)}$$

But it is given that sand is pouring at the rate of $12\ cm^3/sec$.

$$\therefore \frac{dV}{dt} = 12 \quad \text{——— (2)}$$

Now from (1) and (2), we get

$$12 = 36\pi h^2 \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{3\pi h^2}$$

when $h = 4$ then $\frac{dh}{dt} = \frac{1}{3\pi(16)} = \frac{1}{48\pi}$ cm/sec.

Exercise

(1) If a partical is moving in a straight line and its distance from a fixed pt O in the line after t seconds is given by $x = 2-3t+4t^3$ find its acceleration at the end of 2 seconds.

(2) If $S = ut + \frac{1}{2}at^2$, then find velocity at any time t given by $\frac{ds}{dt}$

(3) The velocity of a partical is given by $V = \sqrt{2a(\cos x + x \sin x)}$ show that its accelration is $ax \cos x$

(4) The velocity of a partical is given by $v^2 = a + \frac{2b}{s}$, show that a accelaration is $-\frac{b}{s^2}$

(5) A partical is moving in a straight line according to the formula $s = t^3 - 9t^2 + 3t + 1$, where s is measured in meters and t in seconds. when the velocity is -24 m/sec, find the acceleration.

(6) The distance s in meters described by a partical in t sec. is given by $s = ae^t + be^{-t}$. show that accelvation of the partical at time t is equal to the distance travelled by it up to time t.

(7) A partial moves along a straight line, and its distance from a fixed pt on the line after t seconds from start is given $x = a + bt + ct^2$, where a, b and c are constants. Prove that its moves with a constant acceleration.

(8) How fast is the volume of a ball changing with respect to its radius when the radius is 3m.

(9) The volume of a cube is increasing at a rate of 9 cubic cm. per second. How fast is the surface area increasing when the length of an edge is 10 cm.

(10) A balloon which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the latter is 10 cm.

(11) An edge of a variable cube is increasing at the rate of 3cm/sec. How fast is the volume of the cube increasing when the edge is 10cm long.

(12) The radius of a circle is increasing at 0.7 cm/sec. What is the rate of increase of its circumference when r = 4.9 cm.

(13) A balloon which always remains spherical has a variable diameter $\frac{3}{2}(2x+3)$. Find the rate of change of its volume with respect to x.