Integrals

- Indefinite Integrals

  $\int f(x)dx$

  (this is read as integration of $f(x)$ with respect to x)

- Definite Integrals

  $\int_a^b f(x)\,dx$

  (this is read as integration of $f(x)$ w.r.t. x from a to b)

Note: Here $f(x)$ is integrand, dx is differential of x, a is lower limit of integration, b is upper limit of integration and $\int$ is sign of integration.

Indefinite Integrals: Integration is the inverse process of differentiation.

Let $\frac{d}{dx}(F(x)) = f(x)$

$\Rightarrow \int f(x)dx = F(x)$

Generally, we take $\int f(x)\,dx = F(x) + c$, because

$$\frac{d}{dx}(F(x)+c) = \frac{d}{dx}(F(x)) + \frac{d}{dx}(c) = f(x) + 0$$

$$\Rightarrow \frac{d}{dx}(F(x)+c) = f(x)$$

$$\Rightarrow \int f(x)\,dx = F(x) + c$$

where c is an arbitrary constant and it is known as "constant of integration".

* Some properties of Indefinite Integration:

(i) $\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$

(ii) $\int K.f(x)\,dx = K.\int f(x)\,dx$, where K is constant

(iii) $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

(iv) $\int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$

Standard formulae of Integration:

① $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad ; n \neq -1$

e.g. $\int x^5\,dx = \frac{x^{5+1}}{5+1} + c$

$= \frac{x^6}{6} + c$

② $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \log|x| + c$

③ $\int e^x\,dx = e^x + c$

④ $\int a^x\,dx = \frac{a^x}{\log_e a} + c$

e.g. $\int 7^x\,dx = \frac{7^x}{\log_e 7} + c$

⑤ $\int \sin x\,dx = -\cos x + c$

⑥ $\int \cos x\,dx = \sin x + c$

⑦ $\int \sec^2 x\,dx = \tan x + c$

(8) $\int \sec x \cdot \tan x \, dx = \sec x + c$

(9) $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

(10) $\int \operatorname{cosec} x \cdot \cot x \, dx = -\operatorname{cosec} x + c$

(11) $\int \tan x \, dx = -\log|\cos x| + c$ <u>or</u> $\log|\sec x| + c$

(12) $\int \cot x \, dx = \log|\sin x| + c$

(13) $\int \sec x \, dx = \log|\sec x + \tan x| + c$

(14) $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + c$

(15) $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

(16) $\int \frac{-1}{\sqrt{a^2 - x^2}} \, dx = \cos^{-1}\left(\frac{x}{a}\right) + c$

(17) $\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

(18) $\int \frac{-1}{a^2 + x^2} \, dx = \frac{1}{a} \cot^{-1}\left(\frac{x}{a}\right) + c$

(19) $\int \frac{1}{x\sqrt{x^2 - a^2}} \, dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$

(20) $\int \frac{-1}{x\sqrt{x^2 - a^2}} \, dx = \frac{1}{a} \operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + c$

Q.1. Evaluate the following:

(i) $\int x^2 dx$

(ii) $\int x^3 dx$

(iii) $\int x^{10} dx$

(iv) $\int x^{-5} dx$

(v) $\int x^{-4} dx$

(vi) $\int x^{-3} dx$

(vii) $\int x^{-2} dx$

(viii) $\int x^{-1} dx$

(ix) $\int \sqrt{x}\, dx$

(x) $\int x^{1/3} dx$

(xi) $\int x^{3/2} dx$

(xii) $\int x^{-2/3} dx$

(xiii) $\int x^{-1/2} dx$

(xiv) $\int dx$

(xv) $\int 2\, dx$

(xvi) $\int 3\, dx$

(xvii) $\int (x-5)\, dx$

(xviii) $\int (3x+5)\, dx$

(xix) $\int (-5x^2+10)\, dx$

(xx) $\int (3x^3-2x+5)\, dx$

(xxi) $\int \left(\frac{5}{x}-6x^2-3\right) dx$

(xxii) $\int \left(-\frac{1}{x}+8x+7\right) dx$

(xxiii) $\int \left(\frac{1}{x}+x\right) dx$

(xxiv) $\int \left(2x-\frac{3}{x}+4\right) dx$

(xxv) $\int \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right) dx$

Ans. 1 (i) $\int x^2\,dx = \dfrac{x^{2+1}}{2+1} + c = \dfrac{x^3}{3} + c$ Ans

Ans 1 (iv) $\int x^{-5}\,dx = \dfrac{x^{-5+1}}{-5+1} + c = \dfrac{x^{-4}}{-4} + c$ Ans

Ans. 1 (viii) $\int x^{-1}\,dx = \int \dfrac{1}{x}\,dx = \log|x| + c$ Ans

Ans 1 (x) $\int x^{1/3}\,dx = \dfrac{x^{\frac{1}{3}+1}}{\left(\frac{1}{3}+1\right)} = \dfrac{x^{\frac{1+3}{3}}}{\frac{1+3}{3}} + c = \dfrac{x^{4/3}}{\left(\frac{4}{3}\right)} + c$

$$= \frac{3}{4}x^{4/3} + c$$ Ans

Ans. 1 (xiii) $\int x^{-\frac{1}{2}}\,dx = \dfrac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \dfrac{x^{1/2}}{\left(\frac{1}{2}\right)} + c = 2\sqrt{x} + c$ Ans

Ans 1 (xiv) $\int dx = x + c$ Ans

Ans 1 (xvi) $\int 3\,dx = 3\int dx = 3x + c$

Ans 1 (xviii) $\int (3x+5)\,dx = \int 3x\,dx + \int 5\,dx = 3\int x\,dx + 5\int dx$

$$= 3.\frac{x^2}{2} + 5x + c$$ Ans

Ans. 1 (xxi) $\int \left(\dfrac{5}{x} - 6x^2 - 3\right)dx = \int \dfrac{5}{x}\,dx - \int 6x^2\,dx - \int 3\,dx$

$$= 5\int \frac{1}{x}\,dx - 6\int x^2\,dx - 3\int dx$$

$$= 5\log|x| - \cancel{6}^{\,2} \times \frac{x^3}{\cancel{3}} - 3x + c$$

$$= 5\log|x| - 2x^3 - 3x + c$$ Ans

Ans. 1 (XXV)

$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx$$

$$=\int \sqrt{x}\,dx-\int \frac{1}{\sqrt{x}}\,dx$$

$$=\int x^{1/2}\,dx-\int \frac{1}{x^{1/2}}\,dx$$

$$=\int x^{1/2}\,dx-\int x^{-1/2}\,dx$$

$$=\frac{x^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)}-\frac{x^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)}+c$$

$$=\frac{x^{3/2}}{\left(\frac{3}{2}\right)}-\frac{x^{1/2}}{\left(\frac{1}{2}\right)}+c$$

$$=\frac{2}{3}x^{3/2}-2x^{1/2}+c$$

Q.2. Evaluate the following:

(i) $\int 2^x \, dx$

(ii) $\int (-3^x) \, dx$

(iii) $\int 4^x \, dx$

(iv) $\int 5^x \, dx$

(v) $\int (e^x + 2^x) \, dx$

(vi) $\int (x^a - a^x) \, dx$

(vii) $\int (2 \sin x - 3 \cos x) \, dx$

(viii) $\int (x^5 + 5^x - 3x^2 - 5e^x) \, dx$

(ix) $\int (5 \sec^2 x - 1) \, dx$

(x) $\int (e^x - 3 \sec x \tan x) \, dx$

(xi) $\int \left(\sqrt{x} - \frac{2}{x} + 2^x + x^2\right) dx$

(xii) $\int \left(5 - x + e^x - 3^x + \frac{1}{x}\right) dx$

(xiii) $\int \left(6 \operatorname{cosec}^2 x - 7 + \frac{2}{x}\right) dx$

(xiv) $\int \left(2 \operatorname{cosec} x \cot x - 3^x + x^{-3}\right) dx$

(xv) $\int (\sin x - 5 \cos x + 2 \sec x \tan x) \, dx$

Ans. 2(i) $\int 2^x\,dx = \dfrac{2^x}{\log_e 2} + C$

Ans. 2(ii) $\int (-3^x)\,dx = -\int 3^x\,dx$

$= -\dfrac{3^x}{\log_e 3} + C$

Ans. 2(vi) $\int (x^a - a^x)\,dx = \dfrac{x^{a+1}}{a+1} - \dfrac{a^x}{\log_e a} + C$

Ans. 2(vii) $\int (2\sin x - 3\cos x)\,dx$

$= \int 2\sin x\,dx - \int 3\cos x\,dx$

$= 2\int \sin x\,dx - 3\int \cos x\,dx$

$= 2(-\cos x) - 3(\sin x) + C$

$= -2\cos x - 3\sin x + C$

Ans. 2(viii) $\int (x^5 + 5^x - 3x^2 - 5e^x)\,dx$

$= \int x^5\,dx + \int 5^x\,dx - \int 3x^2\,dx - \int 5e^x\,dx$

$= \dfrac{x^6}{6} + \dfrac{5^x}{\log_e 5} - 3\int x^2\,dx - 5\int e^x\,dx + C$

$= \dfrac{x^6}{6} + \dfrac{5^x}{\log_e 5} - \cancel{3}\cdot\dfrac{x^3}{\cancel{3}} - 5e^x + C$

$= \dfrac{x^6}{6} + \dfrac{5^x}{\log_e 5} - x^3 - 5e^x + C$

Ans. 2(xiii) $\int\left(5-x+e^{x}-3^{x}+\frac{1}{x}\right)dx$

$= \int 5\,dx-\int x\,dx+\int e^{x}dx-\int 3^{x}dx+\int \frac{1}{x}dx$

$= 5x-\frac{x^2}{2}+e^{x}-\frac{3^{x}}{\log_e 3}+\log|x|+c$

Ans 2(xiv) $\int\left(2\operatorname{cosec}x\cot x-3^{x}-x^{-3}\right)dx$

$= 2\int \operatorname{cosec}x\cot x\,dx-\int 3^{x}dx-\int x^{-3}dx$

$= 2(-\operatorname{cosec}(x))-\frac{3^{x}}{\log_e 3}-\frac{x^{-3+1}}{-3+1}+c$

$= -2\operatorname{cosec}x-\frac{3^{x}}{\log_e 3}-\frac{x^{-2}}{-2}+c$

$= -2\operatorname{cosec}x-\frac{3^{x}}{\log_e 3}+\frac{1}{2x^{2}}+c$ Ans

Formulae

① $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a.(n+1)} + c$, provided that $n \neq -1$

② $\int \frac{1}{(ax+b)} \, dx = \frac{\log|ax+b|}{a} + c$

③ $\int e^{ax+b} \, dx = \frac{e^{ax+b}}{a} + c$

④ $\int a^{px+q} \, dx = \frac{a^{px+q}}{p. \log_e a} + c$

⑤ $\int \sin(ax+b) \, dx = \frac{-\cos(ax+b)}{a} + c$

⑥ $\int \cos(ax+b) \, dx = \frac{\sin(ax+b)}{a} + c$

⑦ $\int \sec^2(ax+b) \, dx = \frac{\tan(ax+b)}{a} + c$

⑧ $\int \sec(ax+b).\tan(ax+b) \, dx = \frac{\sec(ax+b)}{a} + c$

⑨ $\int \operatorname{cosec}^2(ax+b) \, dx = -\frac{\cot(ax+b)}{a} + c$

(10) $\int \operatorname{cosec}(ax+b).\cot(ax+b)\,dx = -\frac{\operatorname{cosec}(ax+b)}{a} + c$

(11) $\int \tan(ax+b)\,dx = -\frac{\log|\cos(ax+b)|}{a} + c$

$= \frac{\log|\sec(ax+b)|}{a} + c$

(12) $\int \cot(ax+b)\,dx = \frac{\log|\sin(ax+b)|}{a} + c$

Some examples:

E.1. $\int (7x-12)^{-4}\,dx = \frac{(7x-12)^{-4+1}}{7.(-4+1)} + c = \frac{(7x-12)^{-3}}{-21} + c$

E.2. $\int \frac{1}{2-5x}\,dx = \frac{\log|2-5x|}{-5} + c$

E.3. $\int e^{-2x+1}\,dx = \frac{e^{-2x+1}}{-2} + c$

E.4. $\int 5^{8x+3}\,dx = \frac{5^{8x+3}}{8.\log_e 5} + c$

E.5. $\int \sin(-x+10)\,dx = \frac{-\cos(-x+10)}{-1} + c = \cos(-x+10) + c$

E.6. $\int \operatorname{cosec}^2(10x)\,dx = -\frac{\cot(10x)}{10} + c$

E.7. $\int \tan(7-2x)\,dx = -\frac{\log|\cos(7-2x)|}{-2} + c$

$= \frac{\log|\cos(7-2x)|}{2} + c$

Q.3. Evaluate the following:

(i) $\int (2x+7)^5 dx$

(ii) $\int (5-8x)^3 dx$

(iii) $\int (3x+10)^{-4} dx$

(iv) $\int \frac{1}{(10x+5)} dx$

(v) $\int e^{5x-2} dx$

(vi) $\int e^{8-7x} dx$

(vii) $\int e^{-3x} dx$

(viii) $\int e^{2x} dx$

(ix) $\int 2^{7x-1} dx$

(x) $\int 5^{-x+3} dx$

(xi) $\int 3^{2-4x} dx$

(xii) $\int 4^{5x} dx$

(xiii) $\int \sin(6x-5) dx$

(xiv) $\int \sin 7x \, dx$

(xv) $\int \cos(10x+12) dx$

(xvi) $\int \sin(5-4x) dx$

(xvii) $\int \cos(9-8x) dx$

(xviii) $\int \cos(15x) dx$

(xix) $\int \sec^2(3x+8) dx$

(xx) $\int \sec^2(5-x) dx$

(xxi) $\int \sec^2(2+3x) dx$

(xxii) $\int \sec(2+5x) \tan(2+5x) dx$

(xxiii) $\int \operatorname{cosec}^2(x+8) dx$

(xxiv) $\int \operatorname{cosec}(5-x) . \cot(5-x) dx$

(xxv) $\int \tan(-x+2) dx$

(xxvi) $\int \cot(3x) dx$

(xxvii) $\int \sqrt{2x+5} \, dx$

(xxviii) $\int \frac{1}{(3x-1)^4} dx$

(xxix) $\int \frac{1}{(2x-1)} dx$

(xxx) $\int (x-5)^{1/3} dx$

Ans. 3(ii) $\int (5-8x)^3 dx = \frac{(5-8x)^4}{-8\times 4}+C$

$= \frac{(5-8x)^4}{-32}+C$ Ans

Ans. 3(iv) $\int \frac{1}{(10x+5)}dx = \frac{\log|10x+5|}{10}+c$ Ans

Ans. 3(v) $\int e^{5x-2}dx = \frac{e^{5x-2}}{5}+C$ Ans.

Ans 3(vii) $\int e^{-3x}dx = \frac{e^{-3x}}{-3}+C$ Ans

Ans 3(ix) $\int 2^{7x-1}dx = \frac{2^{7x-1}}{7.\log_e 2}+C$ Ans

Ans 3(xii) $\int 4^{5x}dx = \frac{4^{5x}}{5.\log_e 4}+C$ Ans

Ans. 3(xvi) $\int \sin(5-4x)\,dx = \cancel{-}\frac{\cos(5-4x)}{\cancel{-}4}+C$

$= \cos(5-4x)+C$ Ans

Ans. 3(xvii) $\int \cos(9-8x)\,dx = \frac{\sin(9-8x)}{-8}+C$ Ans

Ans. 3 (xx) $\int \sec^2(5-x)\,dx = \frac{\tan(5-x)}{-1} + C$

$= -\tan(5-x) + C$ Ans.

Ans 3(xxii) $\int \sec(2+5x).\tan(2+5x)\,dx$

$= \frac{\sec(2+5x)}{5} + C$ Ans.

Ans. 3(xxv) $\int \tan(-x+2)\,dx$

$= \frac{\cancel{+}\log|\cos(-x+2)|}{\cancel{+}1} + C$

$= \log|\cos(-x+2)| + C$ Ans

Ans. 3 (xxx) $\int (x-5)^{1/3}\,dx = \frac{(x-5)^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C$

$= \frac{(x-5)^{\frac{1+3}{3}}}{\frac{1+3}{3}} + C$

$= \frac{3}{4}(x-5)^{4/3} + C$ Ans

Note:

① $e^{\log x} = x$

② $\cos 2x = \cos^2 x - \sin^2 x$

③ $\cos^2 x + \sin^2 x = 1$

④ $\tan^2 x = \sec^2 x - 1$

⑤ $\cot^2 x = \operatorname{cosec}^2 x - 1$

⑥ $1 + \cos 2x = 2\cos^2 x$ or $\cos^2 x = \dfrac{1+\cos 2x}{2}$

⑦ $1 - \cos 2x = 2\sin^2 x$ or $\sin^2 x = \dfrac{1-\cos 2x}{2}$

⑧ $\sin^3 x = \dfrac{3\sin x - \sin 3x}{4}$

⑨ $\cos^3 x = \dfrac{\cos 3x + 3\cos x}{4}$

⑩ $2\sin A.\cos B = \sin(A+B) + \sin(A-B)$

⑪ $2\cos A.\sin B = \sin(A+B) - \sin(A-B)$

⑫ $2\cos A.\cos B = \cos(A+B) + \cos(A-B)$

⑬ $2\sin A.\sin B = \cos(A-B) - \cos(A+B)$

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⑭ If $a \overline{)\, b\,}(\, q$

$\quad\; r$

then $\dfrac{b}{a} = q + \dfrac{r}{a}$

Q.4. Evaluate the following:

(i) $\int \left(e^{x \log a} + e^{a \log x} + e^{a \log a}\right) dx$

(ii) $\int \left(\frac{2+3\sin x}{\cos^2 x}\right) dx$

(iii) $\int \frac{\cos 2x}{\sin^2 x \cos^2 x}\, dx$

(iv) $\int \frac{1}{\sin^2 x \cdot \cos^2 x}\, dx$

(v) $\int \tan^2 x\, dx$

(vi) $\int \cot^2 x\, dx$

(vii) $\int \left(\frac{x^4+2}{x^2+1}\right) dx$

(viii) $\int \left(\frac{x^3+2x^2+x+3}{x^2+1}\right) dx$

(ix) $\int \left(\frac{x^5+5x^3+3}{x^2+1}\right) dx$

(x) $\int \frac{1}{\sqrt{5x+3}+\sqrt{5x+2}}\, dx$

(xi) $\int \frac{x+1}{\sqrt{2x-1}}\, dx$

(xii) $\int \frac{1}{\sqrt{x+2}-\sqrt{x+3}}\, dx$

(xiii) $\int \frac{1}{\sqrt{x+1}+\sqrt{x+2}}\, dx$

(xiv) $\int \sin^2 x\, dx$

(xv) $\int \cos^2 x\, dx$

(xvi) $\int \sin^3 x\, dx$

(xvii) $\int \cos^3 x\, dx$

(xviii) $\int 2\sin 4x \cos 3x\, dx$

(xix) $\int \cos 3x \cdot \cos 2x\, dx$

(xx) $\int \sin 4x \cdot \sin 2x\, dx$

Ans. 4(i) $\int (e^{x \log a} + e^{a \log x} + e^{a \log a}) \, dx$

$= \int (e^{\log a^x} + e^{\log x^a} + e^{\log a^a}) \, dx$

(used power property of logarithms)

$= \int (a^x + x^a + a^a) \, dx \qquad (\because e^{\log X} = X)$

$= \int a^x \, dx + \int x^a \, dx + \int a^a \, dx$

$= \dfrac{a^x}{\log_e a} + \dfrac{x^{a+1}}{a+1} + a^a \int dx + c \quad [\because a^a \text{ is a constant}]$

$= \dfrac{a^x}{\log_e a} + \dfrac{x^{a+1}}{a+1} + a^a . x + C$ Ans

Ans. 4(ii) $\int \left(\dfrac{2 + 3 \sin x}{\cos^2 x} \right) dx = \int \left(\dfrac{2}{\cos^2 x} + 3 \dfrac{\sin x}{\cos^2 x} \right) dx$

$= \int \dfrac{2}{\cos^2 x} \, dx + \int 3 \dfrac{\sin x}{\cos x} \cdot \dfrac{1}{\cos x} \, dx$

$= 2 \int \sec^2 x \, dx + 3 \int \tan x . \sec x \, dx$

$= 2 \tan x + 3 \sec x + C$ Ans

Ans 4(iii)

$$\int \frac{\cos 2x}{\sin^2 x \cos^2 x}\, dx$$

$$= \int \left[\frac{\cos^2 x - \sin^2 x}{\sin^2 x . \cos^2 x}\right] dx$$

$$= \int \frac{\cancel{\cos^2 x}}{\sin^2 x . \cancel{\cos^2 x}}\, dx - \int \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x} . \cos^2 x}\, dx$$

$$= \int \frac{1}{\sin^2 x}\, dx - \int \frac{1}{\cos^2 x}\, dx$$

$$= \int \operatorname{cosec}^2 x\, dx - \int \sec^2 x\, dx$$

$$= -\cot x - \tan x + C \quad \text{Ans.}$$

Hint for 4(iv) Use $1 = \sin^2 x + \cos^2 x$ in numerator.

Ans 4(v)

$$\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$$

$$= \int \sec^2 x\, dx - \int 1\, dx$$

$$= \tan x - x + C \quad \text{Ans.}$$

Hint for 4(vi) Use $\cot^2 x = \operatorname{cosec}^2 x - 1$

Ans. 4 (viii) $\int\left(\frac{x^3+2x^2+x+3}{x^2+1}\right)dx$

$$\begin{array}{r|l} x^2+1 & \overline{x^3+2x^2+x+3}\;(x+2 \\ & \underline{-x^3 \qquad -x} \\ & \quad 2x^2+3 \\ & \underline{\quad -2x^2-2} \\ & \qquad 1 \end{array}$$

Here quotient $= x+2$ and remainder $= 1$

$\therefore \frac{x^3+2x^2+x+3}{x^2+1}$

$= (x+2) + \frac{1}{x^2+1}$

$= \int\left(x+2+\frac{1}{x^2+1}\right)dx$

$= \int x\,dx + \int 2\,dx + \int \frac{1}{x^2+1}dx$

$= \frac{x^2}{2} + 2x + \tan^{-1}x + C$ Ans

$\left(\because \int \frac{1}{x^2+1}dx = \tan^{-1}x + C\right)$

Ans. 4 (x) $\int \frac{1}{\sqrt{5x+3}+\sqrt{5x+2}}dx$

$= \int\left[\frac{1}{\sqrt{5x+3}+\sqrt{5x+2}} \times \frac{\sqrt{5x+3}-\sqrt{5x+2}}{\sqrt{5x+3}-\sqrt{5x+2}}\right]dx$

$= \int\left[\frac{\sqrt{5x+3}-\sqrt{5x+2}}{(\sqrt{5x+3})^2-(\sqrt{5x+2})^2}\right]dx$

$= \int\left[\frac{\sqrt{5x+3}-\sqrt{5x+2}}{\cancel{5x}+3-\cancel{5x}-2}\right]dx$

$= \int\left[\frac{\sqrt{5x+3}-\sqrt{5x+2}}{1}\right]dx$

$$= \int (5x+3)^{1/2}\,dx - \int (5x+2)^{1/2}\,dx$$

$$= \frac{(5x+3)^{\frac{1}{2}+1}}{5\cdot\left(\frac{1}{2}+1\right)} - \frac{(5x+2)^{\frac{1}{2}+1}}{5\cdot\left(\frac{1}{2}+1\right)} + c$$

$$= \frac{(5x+3)^{3/2}}{5\times\frac{3}{2}} - \frac{(5x+2)^{3/2}}{5\times\frac{3}{2}} + c$$

$$= \frac{2}{15}(5x+3)^{3/2} - \frac{2}{15}(5x+2)^{3/2} + C$$ Ans

Ans 4 (xi)

$$\int \frac{x+1}{\sqrt{2x-1}}\,dx$$

(Key point is $2x-1$)

Here we are trying to make $2x-1$ in numerator.

$$= \int \frac{2(x+1)}{2\sqrt{2x-1}}\,dx$$

$$= \int \frac{2x+2}{2\sqrt{2x-1}}\,dx$$

$$= \frac{1}{2}\int \left(\frac{2x-1+1+2}{\sqrt{2x-1}}\right)dx$$

$$= \frac{1}{2}\int \left(\frac{(2x-1)+3}{\sqrt{2x-1}}\right)dx$$

$$= \frac{1}{2}\left[\int \frac{2x-1}{\sqrt{2x-1}}\,dx + \int \frac{3}{\sqrt{2x-1}}\,dx\right]$$

$$= \frac{1}{2}\left[\int \frac{(2x-1)}{(2x-1)^{1/2}}\,dx + \int \frac{3}{(2x-1)^{1/2}}\,dx\right]$$

$$= \frac{1}{2}\left[\int (2x-1)^{1-\frac{1}{2}}\,dx + \int 3(2x-1)^{-1/2}\,dx\right]$$

$$= \frac{1}{2}\left[\int (2x-1)^{1/2}\,dx + 3\int (2x-1)^{-1/2}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{(2x-1)^{\frac{1}{2}+1}}{2.\left(\frac{1}{2}+1\right)} + 3\,\frac{(2x-1)^{-\frac{1}{2}+1}}{2.\left(-\frac{1}{2}+1\right)}\right] + c$$

$$= \frac{1}{2}\left[\frac{(2x-1)^{3/2}}{\cancel{2}\times\frac{3}{\cancel{2}}} + \frac{3(2x-1)^{1/2}}{\cancel{2}\times\frac{1}{\cancel{2}}}\right] + C$$

$$= \frac{1}{2}\left[\frac{(2x-1)^{3/2}}{3} + 3(2x-1)^{1/2}\right] + C$$

Ans

Ans 4(xiv) $\int \sin^2 x \, dx$

$$= \int \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \int \frac{1}{2} dx - \int \frac{\cos 2x}{2} dx$$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x \, dx$$

$$= \frac{1}{2}.x - \frac{1}{2}.\frac{\sin 2x}{2} + C$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C \quad \text{Ans}$$

Hint for 4(xv) Put $\cos^2 x = \frac{1+\cos 2x}{2}$

Hint for 4(xvi) Put $\sin^3 x = \frac{3\sin x - \sin 3x}{4}$

Ans 4(xvii) $\int \cos^3 x \, dx = \int \left(\frac{\cos 3x + 3\cos x}{4}\right) dx$

$$= \frac{1}{4}\left[\frac{\sin 3x}{3} + 3(\sin x)\right] + C$$

$$= \frac{\sin 3x}{12} + \frac{3\sin x}{4} + C \quad \text{Ans}$$

Ans 4(xviii) $\int 2\sin 4x.\cos 3x\,dx$

$= \int [\sin(4x+3x)+\sin(4x-3x)]\,dx$

[used $2\sin A\cos B = \sin(A+B)+\sin(A-B)$]

$= \int (\sin 7x + \sin x)\,dx$

$= -\frac{\cos 7x}{7} - \cos x + C$ Ans

Ans. 4(xx) $\int \sin 4x \sin 2x\,dx$

$= \frac{1}{2}\int 2\sin 4x \sin 2x\,dx$

$= \frac{1}{2}\int [\cos(4x-2x) - \cos(4x+2x)]\,dx$

($\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)$)

$= \frac{1}{2}\int (\cos 2x - \cos 6x)\,dx$

$= \frac{1}{2}\left[\frac{\sin 2x}{2} - \frac{\sin 6x}{6}\right] + C$

$= \frac{\sin 2x}{4} - \frac{\sin 6x}{12} + C$ Ans.

Integration by parts: If u and v both are functions of x, then

$$\int u.v\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\cdot\int v\,dx\right]dx$$

Note: Here, we have taken u as first function and v as second function. Further, in the selection of 1st function and 2nd function, we can take the help of following:

I L A T E

- I → Inverse Trignometric Functions (e.g. $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc.)
- L → Logarithmic Functions (e.g. $\log x$, $\log(x+1)$, $\log 5x$ etc.)
- A → Algebraic Functions (e.g. any constant, any polynomial, $2, 5, 1, x, x^2, x^3, 2x+1, 3-x$ etc.)
- T → Trignometric Functions (e.g. $\sin x$, $\cos x$, $\tan x$, $\cot x$ etc.)
- E → Exponential Functions (e.g. e^x, 2^x, 3^x, 4^x etc.)

If the integrand contains two functions of different types, then take that function as first which appears first in the word ILATE.

e.g. $\int x.\log x\,dx$

Here x is algebraic function and $\log x$ is logarithmic function. So, we will take $\log x$ as first function and x as 2nd function to solve the given problem.

Q.5. Evaluate the following:

(i) $\int x e^x \, dx$

(ii) $\int x \sin x \, dx$

(iii) $\int x \cdot \log x \, dx$

(iv) $\int \cos x \cdot x \, dx$

(v) $\int \log x \cdot x^2 \, dx$

(vi) $\int \log x \, dx$

(vii) $\int x e^{2x} \, dx$

(viii) $\int x e^{-x} \, dx$

(ix) $\int x \sin 2x \, dx$

(x) $\int x \cdot \cos 2x \, dx$

(xi) $\int x^2 e^{-x} \, dx$

(xii) $\int x \cdot \log 2x \, dx$

(xiii) $\int x e^{5x} \, dx$

(xiv) $\int x^2 e^x \, dx$

(xv) $\int x^2 \cdot \sin x \, dx$

Ans 5 (i) $\int \underset{I}{x}\, \underset{II}{e^x}\, dx$

$u \to x$
$v \to e^x$

$= x\int e^x dx - \int\left[\frac{dx}{dx}\cdot \int e^x dx\right]dx$

$= x e^x - \int [1.e^x]\, dx + c$

$= x e^x - \int e^x dx + c$

$= x e^x - e^x + c$ Ans.

Ans. 5 (ii) $\int \underset{I}{x}.\, \underset{II}{\sin x}\, dx$

$u \to x$
$v \to \sin x$

$= x\int \sin x\, dx - \int\left[\frac{dx}{dx}\cdot \int \sin x\, dx\right]dx$

$= x(-\cos x) - \int 1.(-\cos x)\, dx + C$

$= -x\cos x + \int \cos x\, dx + C$

$= -x\cos x + \sin x + C$ Ans.

Ans. 5 (iii) $\int \underset{II}{x}.\, \underset{I}{\log x}\, dx$

$u \to \log x$
$v \to x$

$= \log x \int x\, dx - \int\left[\frac{d}{dx}(\log x).\ \int x\, dx\right]dx$

$= \log x.\frac{x^2}{2} - \int\left(\frac{1}{\cancel{x}} \times \frac{x^{\cancel{2}}}{2}\right)dx + C$

$= \frac{x^2}{2}\log x - \frac{1}{2}\int x\, dx + C$

$= \frac{x^2}{2}\log x - \frac{x^2}{4} + C$ Ans.

Ans 5 (vi) $\int \log x \, dx$

$= \int \underset{I}{\log x} \, . \, \underset{II}{1} \, dx$ $u \to \log x$, $v \to 1$

$= \log x . \int 1 \, dx - \int \left[\frac{d}{dx}(\log x) . \int 1 \, dx \right] dx$

$= \log x . x - \int \left[\frac{1}{\cancel{x}} \times \cancel{x} \right] dx + c$

$= x . \log x - \int 1 \, dx + c$

$= x \log x - x + c$ Ans

Ans 5 (viii) $\int x e^{-x} dx$ $u \to x$, $v \to e^{-x}$

$= x \int e^{-x} dx - \int \left[\frac{dx}{dx} \int e^{-x} dx \right] dx$

$= x . \frac{e^{-x}}{-1} - \int \left[1 . \frac{e^{-x}}{-1} \right] dx + c$

$= -x e^{-x} + \int e^{-x} dx + c$

$= -x e^{-x} + \frac{e^{-x}}{-1} + c = -x e^{-x} - e^{-x} + c$ Ans

Ans. 5(x) $\int x . \cos 2x \, dx$ $u \to x$, $v \to \cos 2x$

$= x \int \cos 2x \, dx - \int \left[\frac{dx}{dx} \int \cos 2x \, dx \right] dx$

$= x . \frac{\sin 2x}{2} - \int \left[1 . \frac{\sin 2x}{2} \right] dx + c$

$= \frac{x \sin 2x}{2} - \frac{1}{2} \int \sin 2x \, dx + c = \frac{x \sin 2x}{2} - \frac{1}{2} \left(-\frac{\cos 2x}{2} \right) + c$

$= \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + c$ Ans

Ans. 5 (xiv)

$$\int x^2 e^x \, dx \qquad \begin{matrix} u \to x^2 \\ v \to e^x \end{matrix}$$

$$= x^2 \int e^x dx - \int \left[\frac{d}{dx}(x^2) \cdot \int e^x dx \right] dx$$

$$= x^2 \cdot e^x - \int [2x \, e^x] \, dx + C$$

$$= x^2 e^x - 2 \int \underset{u}{x} \, \underset{v}{e^x} \, dx + C$$

$$= x^2 e^x - 2 \left[x \int e^x dx - \int \left[\frac{dx}{dx} \cdot \int e^x dx \right] dx \right] + C$$

$$= x^2 e^x - 2 \left[x e^x - \int [1 \cdot e^x] \, dx \right] + C$$

$$= x^2 e^x - 2 \left[x e^x - e^x \right] + C$$

$$= x^2 e^x - 2x e^x + 2e^x + C$$ Ans

Ans. 5 (xv)

$$\int x^2 \cdot \sin x \, dx \qquad \begin{matrix} u \to x^2 \\ v \to \sin x \end{matrix}$$

$$= x^2 \int \sin x \, dx - \int \left[\frac{d}{dx}(x^2) \cdot \int \sin x \, dx \right] dx$$

$$= x^2 \cdot (-\cos x) - \int [2x \cdot (-\cos x)] \, dx + C$$

$$= -x^2 \cos x + 2 \int x \cos x \, dx + C$$

$$= -x^2 \cos x + 2 \left[x \int \cos x \, dx - \int \left[\frac{dx}{dx} \cdot \int \cos x \, dx \right] dx \right] + C$$

$$= -x^2 \cos x + 2 \left[x \sin x - \int 1 \cdot \sin x \, dx \right] + C$$

$$= -x^2 \cos x + 2 \left[x \sin x - (-\cos x) \right] + C$$

$$= -x^2 \cos x + 2 \left[x \sin x + \cos x \right] + C$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C$$ Ans.

Integral of the form $\int e^x(f(x)+f'(x))\,dx$:

Prove that $\int e^x(f(x)+f'(x))\,dx = e^x f(x) + C$

Sol: $\int e^x(f(x)+f'(x))\,dx = \int \underset{\mathrm{II}}{e^x}\,\underset{\mathrm{I}}{f(x)}\,dx + \int e^x f'(x)\,dx$

$$= f(x)\int e^x dx - \int\left[\frac{d}{dx}(f(x))\int e^x dx\right]dx + \int e^x f'(x)\,dx$$

$$= f(x).e^x - \cancel{\int f'(x).e^x dx} + \cancel{\int e^x f'(x)\,dx} + C$$

$$= e^x.f(x) + C \quad \text{Ans}$$

Q.6. Evaluate the following:

(i) $\int e^x(\sin x + \cos x)\,dx$ (ii) $\int e^x(\tan x + \sec^2 x)\,dx$

(iii) $\int e^x(x^2+2x)\,dx$ (iv) $\int e^x(\sin 2x + 2\cos 2x)\,dx$

(v) $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx$ (vi) $\int e^x(x^3+3x^2)\,dx$

(vii) $\int e^x\left[\frac{x}{(x+1)^2}\right]dx$ (viii) $\int e^x(\sin mx + m\cos mx)\,dx$

Ans. 6 (i) $\int e^x(\sin x + \cos x)\,dx$

$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{\sin x}\, dx + \int e^x \cos x\, dx$

$= \sin x \int e^x dx - \int\left[\frac{d(\sin x)}{dx} \cdot \int e^x dx\right]dx + \int e^x \cos x\, dx$

$= \sin x . e^x - \cancel{\int(\cos x . e^x)dx} + \cancel{\int e^x . \cos x\, dx} + C$

$= \sin x . e^x + C$ Ans.

Ans. 6 (iii) $\int e^x(x^2 + 2x)\,dx$

$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{x^2}\, dx + \int e^x . 2x\, dx$

$= x^2 \int e^x dx - \int\left[\frac{d}{dx}(x^2) . \int e^x dx\right]dx + \int 2x e^x dx$

$= x^2 . e^x - \cancel{\int(2x . e^x)dx} + \cancel{\int 2x e^x dx} + C$

$= x^2 e^x + C$ Ans.

Ans. 6 (iv) $\int e^x(\sin 2x + 2\cos 2x)\,dx$

$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{\sin 2x}\, dx + \int e^x . 2\cos 2x\, dx$

$= \sin 2x . \int e^x dx - \int\left[\frac{d}{dx}(\sin 2x) . \int e^x dx\right]dx + \int e^x . 2\cos 2x\, dx$

$= \sin 2x . e^x - \int\left[\cos 2x . \frac{d}{dx}(2x) . e^x\right]dx + \int 2\cos 2x . e^x dx + C$

$= \sin 2x . e^x - \int\left[\cos 2x \times 2 \times 1 . e^x\right]dx + \int 2\cos 2x\, e^x dx + C$

$= \sin 2x . e^x - \cancel{\int 2\cos 2x\, e^x dx} + \cancel{\int 2\cos 2x\, e^x dx} + C$

$= \sin 2x . e^x + C$ Ans.

Ans. 6(vii) $\int e^x\left[\frac{x}{(x+1)^2}\right]dx$

$$= \int e^x\left[\frac{(x+1)-1}{(x+1)^2}\right]dx$$

$$= \int e^x\left[\frac{x+1}{(x+1)^2} - \frac{1}{(x+1)^2}\right]dx$$

$$= \int e^x\left[\frac{1}{(x+1)} - \frac{1}{(x+1)^2}\right]dx$$

$$= \int e^x \cdot \frac{1}{(x+1)}\,dx - \int e^x \cdot \frac{1}{(x+1)^2}\,dx$$

$$= \frac{1}{(x+1)} \cdot \int e^x dx - \int\left[\frac{d}{dx}\left(\frac{1}{(x+1)}\right) \cdot \int e^x dx\right]dx - \int e^x \cdot \frac{1}{(x+1)^2}\,dx$$

$$= \frac{1}{(x+1)} \cdot e^x - \int\left[\frac{d}{dx}\left((x+1)^{-1}\right) \cdot e^x\right]dx - \int e^x \cdot \frac{1}{(x+1)^2}\,dx + c$$

$$= \frac{1}{(x+1)} \cdot e^x - \int\left[(-1)(x+1)^{-2} \cdot \frac{d}{dx}(x+1) \cdot e^x\right]dx - \int e^x (x+1)^{-2} dx + c$$

$$= \frac{e^x}{(x+1)} + \int (x+1)^{-2} \cdot (1+0) \cdot e^x dx - \int e^x (x+1)^{-2} dx + c$$

$$= \frac{e^x}{(x+1)} + \cancel{\int (x+1)^{-2} \cdot e^x dx} - \cancel{\int e^x (x+1)^{-2} dx} + c$$

$$= \frac{e^x}{(x+1)} + C$$

Ans