INDICES

A power, or an index, is used to write a product of numbers very compactly. For example: We write the expression

$4 \times 4 \times 4 \times 4 \times 4$ as 4^5 and we read this as ‘four to the power five’.

Similarly, $a \times a \times a = a^3$, we read it as ‘$a$ to the power three’ or ‘a cubed’.

In the expression a^b, b is the index and a is the base.

Exercise:

Q.1 Write the following expressions in compact form by using an index:

(i) $3 \times 3 \times 3 \times 3 \times 3 \times 3$

(ii) $b \times b \times b \times b$

(iii) $(xy) \times (xy) \times (xy)$

Answers 1:

(i) 3^6

(ii) b^4

(iii) $(xy)^3$

The laws of indices

First law: $\boldsymbol{a^m \times a^n = a^{m+n}}$

When expressions with the same base are multiplied, the indices are added.

For example:

$5^7 \times 5^3 = 5^{7+3} = 5^{10}$

Second Law: $\boldsymbol{\frac{a^m}{a^n} = a^{m-n}}$

When expressions with the same base are divided, the indices are subtracted.

For example:

$\frac{5^7}{5^3} = 5^{7-3} = 5^4$

Third law: $\boldsymbol{(a^m)^n = a^{mn}}$

For example:

$(7^3)^4 = 7^{3\times4} = 7^{12}$

Some important points to remember:

$a^0 = 1$ and $a^1 = a$.

Exercise:

Q.2 Choose the appropriate law of indices to write compact form of following expression:

(i) $4^4 \times 4^5$

(ii) $\frac{8^6}{8^4}$

(iii) $(5^2)^6$

(iv) $\frac{(xy)^4}{(xy)^3}$

Answers 2:

(i) 4^9

(ii) 8^2

(iii) 5^{12}

(iv) (xy)

FACTORISATION

Factorisation is the process to write an expression into product of two or more factors.

For example: $x^2 - 1 = (x - 1)(x + 1)$

Some Standard Results:

(i) $x^2 - y^2 = (x - y)(x + y)$

(ii) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

(iii) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

(iv) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

Q.3 Factorise the following expressions:

(i) $6xy - 9x^2$

(ii) $8a^3b + 4a^2b^2$

(iii) $3(x - y)^2 + 2(x - y)$

(iv) $14a^2bc^3 - 21ab^2c^2$

Answer 3:

(i) $6xy - 9x^2 = 3x(2y - 3x)$

(ii) $8a^3b + 4a^2b^2 = 4a^2b(2a + b)$

(iii) $3(x - y)^2 + 2(x - y) = (x - y)[3(x - y) + 2] = (x - y)(3x - 3y + 2)$

(iv) $14a^2bc^3 - 21ab^2c^2 = 7abc^2(2ac - 3b)$

Q.4 Factorise the following expressions by using standard results:

(i) $x^2 - 25$

(ii) $8x^3 - 125y^3$

(iii) $a^3 + 64b^3$

(iv) $x^3 + 8y^3 + 27z^3 - 18xyz$

Answer 4:

(i) $x^2 - 25 = x^2 - 5^2 = (x - 5)(x + 5)$

$[used\ x^2 - y^2 = (x - y)(x + y)]$

(ii) $8x^3 - 125y^3 = (2x)^3 - (5y)^3$

$= (2x - 5y)(4x^2 + 10xy + 25y^2)$

$[used\ x^3 - y^3 = (x - y)(x^2 + xy + y^2)]$

(iii) $a^3 + 64b^3 = a^3 + (4b)^3$

$= (a + 4b)(a^2 + 4ab + 16b^2)$

$[used\ x^3 + y^3 = (x + y)(x^2 - xy + y^2)]$

(iv) $x^3 + 8y^3 + 27z^3 - 18xyz$

$= x^3 + (2y)^3 + (3z)^3 - 3 \times x \times 2y \times 3z$

$= (x + 2y + 3z)(x^2 + 4y^2 + 9z^2 - 2xy - 12yz - 6zx)$

$[\ used\ x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)]$

EXPANSIONS

Expansion is the process to write an expression into expanded form.

For example: $(x-1)(x+1) = x^2 - 1$

Some Standard Results:

(i) $(x+y)^2 = x^2 + y^2 + 2xy$

(ii) $(x-y)^2 = x^2 + y^2 - 2xy$

(iii) $(x+y)^3 = x^3 + y^3 + 3x^2y + 3xy^2$

(iv) $(x-y)^3 = x^3 - y^3 - 3x^2y + 3xy^2$

(v) $(x+y)(x-y) = x^2 - y^2$

(vi) $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

Q.5 Express the following:

(i) $(2x+3y)^2$

(ii) $(3a-b)^2$

(iii) $\left(x+\frac{1}{x}\right)^3$

(iv) $(x-2y)^3$

(v) $(7a+8b)(7a-8b)$

(vi) $(x+2y+3z)^2$

Answer 5:

(i) $(2x+3y)^2 = 4x^2 + 9y^2 + 2 \times 2x \times 3y$

$= 4x^2 + 9y^2 + 12xy$

$[used\ (x+y)^2 = x^2 + y^2 + 2xy]$

(ii) $(3a-b)^2 = 9a^2 + b^2 - 2 \times 3a \times b$

$= 9a^2 + b^2 - 6ab$

$[used\ (x-y)^2 = x^2 + y^2 - 2xy]$

(iii) $\left(x+\frac{1}{x}\right)^3 = x^3 + \left(\frac{1}{x}\right)^3 + 3x^2 \times \frac{1}{x} + 3x \times \left(\frac{1}{x}\right)^2$

$= x^3 + \frac{1}{x^3} + 3x + \frac{3}{x}$

$[used\ (x+y)^3 = x^3 + y^3 + 3x^2y + 3xy^2]$

(iv) $(x-2y)^3 = x^3 - 8y^3 - 3x^2 \times 2y + 3x \times 4y^2$

$= x^3 - 8y^3 - 6x^2y + 12xy^2$

$[\,used\ (x-y)^3 = x^3 - y^3 - 3x^2y + 3xy^2]$

(v) $(7a+8b)(7a-8b) = (7a)^2 - (8b)^2$

$= 49a^2 - 64b^2$

$[used\ (x+y)(x-y) = x^2 - y^2]$

(vi) $(x+2y+3z)^2 = x^2 + (2y)^2 + (3z)^2 + 2x \times 2y + 2 \times 2y \times 3z + 2 \times 3z \times x$

$= x^2 + 4y^2 + 9z^2 + 4xy + 12yz + 6zx$

$[used\ (x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx]$

Complex Numbers

Complex Number:

A number of the form $a + ib$, where a and b are real numbers and $i^2 = -1$, is called complex number. Here $\boldsymbol{a}$ is called the real part and $\boldsymbol{b}$ is called the imaginary part of $\boldsymbol{a + ib}$.
If $a = 0$, the number ib is said to be a purely imaginary number and if $b = 0$, the number a is real.
The complex numbers are usually denoted by Z, i.e., $Z = a + ib$

Properties of Complex Number:

(i) Two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$.
For example:
If $2x - iy = 3 + 4i$, then $2x = 3$ and $-y = 4$.

(ii) If any complex number becomes zero then its real and imaginary parts will separately become zero.

Some Basic Algebraic Operation on Complex Numbers:

If $Z_1 = a_1 + ib_1$ and $Z_2 = a_2 + ib_2$, then
(i) **Addition:**
$Z_1 + Z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$

(ii) **Subtraction:**
$Z_1 - Z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2)$

(iii) **Multiplication:**
$Z_1.Z_2 = (a_1 + ib_1).(a_2 + ib_2)$
$= (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)$

(iv) **Division:**
$\frac{Z_1}{Z_2} = \frac{(a_1 + ib_1)}{(a_2 + ib_2)}$
Multiply Numerator and denominator by the number$(a_2 - ib_2)$ in order to make the denominator real.

$$\frac{Z_1}{Z_2} = \frac{(a_1 + ib_1)}{(a_2 + ib_2)} \times \frac{(a_2 - ib_2)}{(a_2 - ib_2)}$$
$$= \frac{(a_1a_2 + b_1b_2) + i(b_1a_2 - a_1b_2)}{\left(a_2{}^2 + b_2{}^2\right)}$$

Example 1. If $Z_1 = 2 + 3i$ and $Z_2 = 3 - 4i$, find $Z_1 + Z_2$, $Z_1 - Z_2$, $Z_1.Z_2$ and $\frac{Z_1}{Z_2}$. **Solution:**
$Z_1 + Z_2 = (2 + 3i\ ) + (3 - 4i) = 5 - i.$
$Z_1 - Z_2 = (2 + 3i\ ) - (3 - 4i) = -1 + 7i.$
$Z_1.Z_2 = (2 + 3i\ ).(3 - 4i) = 18 + i.$
$$\frac{Z_1}{Z_2} = \frac{2 + 3i}{3 - 4i} = \frac{(6 - 12) + (9 + 8)i}{9 + 16} = -\frac{6}{25} + \frac{17}{25}i$$

Example 2. Express $-2i + 3(5-1) - 6$ in the form of $a + ib$.
Solution: $-2i + 3(5-i) - 6 = -2i + 15 - 3i - 6$
$= 9 - 5i$

Conjugate of a complex number:

If $Z = a + ib$ is a complex number, then conjugate of Z is given by $a - ib$. Also conjugate of Z is represented by $\bar{Z} = a - ib$.

Theorem(without proof): If Z_1 and Z_2 are two complex numbers, then
(i) $\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$
(ii) $\overline{Z_1 - Z_2} = \overline{Z_1} - \overline{Z_2}$
(iii) $\overline{Z_1 . Z_2} = \overline{Z_1} . \overline{Z_2}$
(iv) $\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\overline{Z_1}}{\overline{Z_2}}$, provided that $Z_2 \neq 0$.

Example 3. Find the conjugate of the following:
(i) $Z_1 = -2 + 3i$ (ii) $Z_2 = -5i$ (iii) $Z_3 = -7$ (iv) $Z_4 = 5 - 3i$
Solution:
(i) Given that $Z_1 = -2 + 3i$
Therefore, $\overline{Z_1} = \overline{(-2 + 3i)} = -2 - 3i$
(ii) Given that $Z_2 = -5i$
Therefore, $\overline{Z_2} = \overline{(-5i)} = 5i$
(iii) Given that $Z_3 = -7$
Therefore, $\overline{Z_3} = \overline{(-7)} = -7$
(iv) Given that $Z_4 = 5 - 3i$
Therefore, $\overline{Z_4} = \overline{(5 - 3i)} = 5 + 3i$

Example 4. If $Z = 2 + 3i$, evaluate $Z \times \bar{Z}$.
Solution: Given that $Z = 2 + 3i$
Therefore, $\bar{Z} = 2 - 3i$
Now, $Z \times \bar{Z} = (2 + 3i)(2 - 3i)$
$= 4 + 9 = 13$

Modulus of a Complex Number:

If $Z = a + ib$ is a complex number, then modulus of Z is given by $\sqrt{a^2 + b^2}$. Also, modulus of Z is represented by $|Z| = \sqrt{a^2 + b^2}$.

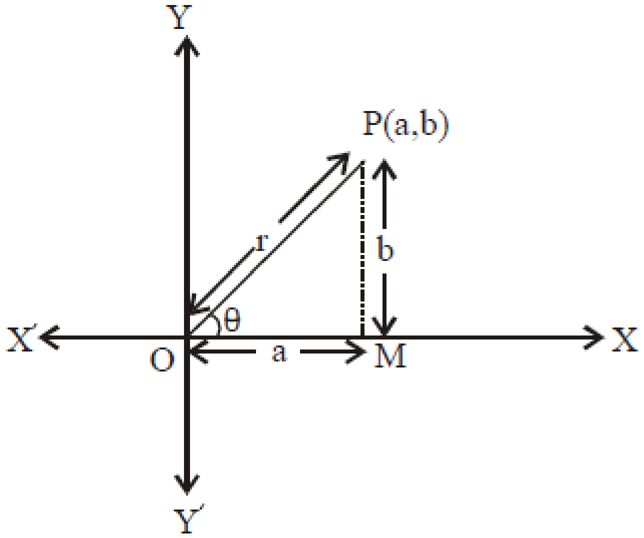
Note:
(i) $Z.\bar{Z} = |Z|^2 = a^2 + b^2$

Example 5. Find the modulus of $Z = 4 - 3i$.
Solution:
Given that $Z = 4 - 3i$
Therefore, $|Z| = \sqrt{4^2 + (-3)^2}$
$= \sqrt{16 + 9}$
$= \sqrt{25} = 5$

Example 6. If $Z = 2 + i$, verify that $Z.\bar{Z} = |Z|^2$.
Solution:
Given that $Z = 2 + i$
Therefore, $\bar{Z} = 2 - i$,
$Z.\bar{Z} = (2 + i)(2 - i) = 4 + 1 = 5$,
$|Z| = \sqrt{2^2 + 1^2} = \sqrt{5}$,
$|Z|^2 = 5$
Hence it is verified that $Z.\bar{Z} = |Z|^2$.

Argument and Polar Form of a complex number:

Let P(a, b) represent the complex number Z=a+ib, where a and b are real numbers, and OP makes an angle θ with the positive direction of X-axis.
Draw PM ⊥ OX



Let OP = r
In ΔOMP
OM = a
MP = b
Therefore, r cosθ = a
and r sinθ = b
Then, Z can be written as Z = r (cosθ + i sinθ)
where r = $\sqrt{a^2 + b^2}$
and tanθ = $\frac{b}{a}$
i.e. θ= $tan^{-1}\left(\frac{b}{a}\right)$
This new form of Z is known as the polar form of the complex number Z, and r and θ are respectively called the modulus and the argument of the complex number.

Example 7. Express the complex number $Z = 1 + i$ in polar form.
Solution:
Given that $Z = 1 + i$
Here, we have $a = 1$ and $b = 1$

Therefore, $r = \sqrt{1^2 + 1^2} = \sqrt{2}$
and $tan\theta = \frac{b}{a} = 1$
i.e. $\theta = tan^{-1}(1) = \frac{\pi}{4}$
Hence, $Z = 1 + i = \sqrt{2}(cos\frac{\pi}{4} + isin\frac{\pi}{4})$

Example 8. Express the complex number $Z = \sqrt{3} + i$ in polar form.
Solution:
Given that $Z = \sqrt{3} + i$
Here, we have $a = \sqrt{3}$ and $b = 1$
Therefore, $r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$
and $tan\theta = \frac{1}{\sqrt{3}}$
i.e. $\theta = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$
Hence, $Z = \sqrt{3} + i = 2(cos\frac{\pi}{6} + isin\frac{\pi}{6})$

Cartesian or Rectangular Form of a complex number:

If $Z = r(cos\theta + i\,sin\theta)$,
then we can express Z in the Cartesian form i.e. $a + ib$, where $a = r\,cos\theta$ and $b = r\,sin\theta$.

Example 9. Express the complex number $Z = \sqrt{2}(cos\frac{\pi}{4} + isin\frac{\pi}{4})$ in Cartesian form.
Solution:
Given that $Z = \sqrt{2}(cos\frac{\pi}{4} + isin\frac{\pi}{4})$
Here, we have $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$
Therefore, $a = r\,cos\theta = \sqrt{2}\,.cos\frac{\pi}{4} = \sqrt{2}\,.\frac{1}{\sqrt{2}} = 1$
and $b = r\,sin\theta = \sqrt{2}\,.sin\frac{\pi}{4} = \sqrt{2}\,.\frac{1}{\sqrt{2}} = 1$
Hence, $Z = \sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right) = 1 + i$

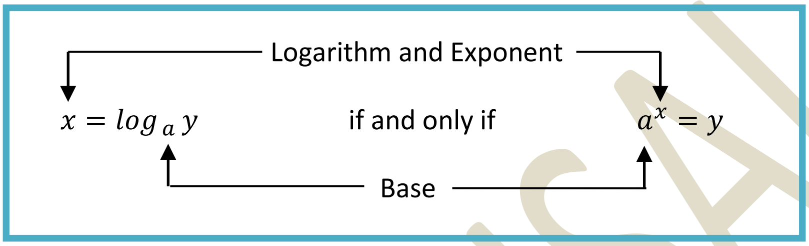
Example 10. Express the complex number $Z = 2(cos\frac{\pi}{6} + isin\frac{\pi}{6})$ in Cartesian form.
Solution:
Given that $Z = 2(cos\frac{\pi}{6} + isin\frac{\pi}{6})$
Here, we have $r = 2$ and $\theta = \frac{\pi}{6}$
Therefore, $a = r\,cos\theta = 2\,.cos\frac{\pi}{6} = 2\,.\frac{\sqrt{3}}{2} = \sqrt{3}$
and $b = r\,sin\theta = 2\,.sin\frac{\pi}{6} = 2\,.\frac{1}{2} = 1$
Hence, $Z = 2(cos\frac{\pi}{6} + isin\frac{\pi}{6}) = \sqrt{3} + i$

LOGARITHM

Definition of Logarithm: If y and a are positive real numbers($a \neq 1$), then $x = log_a y$ if and only if $a^x = y$.

The notation $log_a y$ is read as "log to the base a of y". In the equation $x = log_a y$, x is known as the **logarithm**, a is the **base** and y is the **argument**.

Note: 1. The above definition indicates that a logarithm is an exponent.



2. Logarithm of a number may be negative but the argument of logarithm must be positive. The base must also be positive and not equal to 1.

3.

Logarithmic Form	Exponential Form
$x = log_a y$	$a^x = y$

4. Logarithm of zero doesn't exist.

5. Logarithms of negative real numbers are not defined in the system of real numbers.

6. Log to the base "10" is called Common Logarithm and Log to the base "e" is called Natural Logarithm.($e = 2.7182818284 \dots$)

7. If base of logarithm is not given, we'll consider it Natural Logarithm.

Some examples of logarithmic form and their corresponding exponential form:

S. No.	Logarithmic form	Exponential form
1	$5 = log_2 32$	$2^5 = 32$
2	$4 = log_3 81$	$3^4 = 81$
3	$3 = log_5 125$	$5^3 = 125$
4	$4 = log_{10} 10000$	$10^4 = 10000$
5	$-2 = log_7 \left(\frac{1}{49}\right)$	$7^{-2} = \frac{1}{49}$
6	$0 = log_e 1$	$e^0 = 1$

Why do we study logarithms: Sometimes multiplication, subtraction and exponentiation become so lengthy and tedious to solve. Logarithms covert the problems of multiplication into addition, division into subtraction and exponentiation into multiplication, which are easy to solve.

Properties of Logarithms: If a, b and c are positive real numbers, $a \neq 1$ and n is any real number, then

1. **Product property:** $\boldsymbol{log_a(b.c) = log_a b + log_a c}$

   For example: $log_{10}(187) = log_{10}(11 \times 17) = log_{10}11 + log_{10}17$

2. **Quotient property:** $\boldsymbol{log_a\left(\frac{b}{c}\right) = log_a b - log_a c}$

   For example: $log_7\left(\frac{51}{7}\right) = log_7 51 - log_7 7$
3. **Power property:** $\boldsymbol{log_a\, b^n = n.log_a\, b}$

   For example: $log_{10}(10000) = log_{10}(10^4) = 4.log_{10}10$
4. **One to One property:** $\boldsymbol{log_a b = log_a c}$ **if and only if** $\boldsymbol{b = c}$.

   For example: If $log_{10}(a) = log_{10}(15)$ then $a = 15$.
5. $\boldsymbol{log_a 1 = 0}$

   For example: $log_{10}(1) = 0$, $log_2(1) = 0$, $log_e(1) = 0$ etc.
6. $\boldsymbol{log_a a = 1}$

   For example: $log_{10}(10) = 1$, $log_e(e) = 1$ etc.
7. $\boldsymbol{log_a a^n = n}$

   For example: $log_{10}10^4 = 4$
8. $\boldsymbol{a^{log_a(n)} = n}$**, where** $\boldsymbol{n} > 0$

   For example: $2^{log_2(8)} = 8$
9. **Change of base property:** $\boldsymbol{log_a\, b = \frac{log(b)}{log(a)} = \frac{log_c(b)}{log_c(a)}}$ **provided that** $\boldsymbol{c \neq 1}$.

   For example: $log_2(3) = \frac{log_{10}(3)}{log_{10}(2)}$ (Here we changed the base to 10)

Some solved problems:

Q.1. Convert the following exponential forms into logarithmic forms:

(i) $9^3 = 729$ **(ii)** $4^{-2} = 0.0625$

Ans. (i) Given that $9^3 = 729$

$\Rightarrow \quad log_9 729 = 3 \qquad (by\ definition)$

which is required logarithmic form.

OR

Given that $9^3 = 729$

Taking logarithm on both sides, we get

$log 9^3 = log 729$

$\Rightarrow \quad 3\ log9 = log729$ $\quad (used\ power\ property)$

$\Rightarrow \quad 3 = \frac{log729}{log9}$

$\Rightarrow \quad 3 = log_9 729$ $\quad \left(used\ log_a b = \frac{log(b)}{log(a)}\right)$

which is required logarithmic form.

(ii) Given that $4^{-2} = 0.0625$

$\Rightarrow \quad log_4 0.0625 = -2$

which is required logarithmic form.

Q.2. Convert the following logarithmic forms into exponential forms:

(i) $log_\pi 1 = 0$ **(ii)** $log_{10} 0.01 = -2$

Ans. (i) Given that $log_\pi 1 = 0$

$\Rightarrow \quad \pi^0 = 1$

which is required exponential form.

(v) Given that $log_{10} 0.01 = -2$

$\Rightarrow \quad 10^{-2} = 0.01$

which is required exponential form.

Q.3. Evaluate the following:

(i) $log_2(8 \times 16)$ **(ii)** $log_{10}\left(\frac{1}{10}\right)^8$

Ans. (i) Given expression is

$log_2(8 \times 16) = log_2(8) + log_2(16)$ $\quad (used\ product\ property)$

$= log_2 2^3 + log_2 2^4$

$= 3. log_2 2 + 4. log_2 2$ $\quad (used\ power\ property)$

$= 3 + 4 = 7$ $\quad (used\ log_a a = 1)$

which is required solution.

OR

Given expression is

$log_2(8 \times 16) = log_2(128)$

$= log_2 2^7 = 7$ $\quad (used\ log_a a^n = n)$

which is required solution.

(iv) Given expression is

$$log_{10}\left(\frac{1}{10}\right)^8 = log_{10}\frac{1}{10^8}$$

$$= log_{10}1 - log_{10}10^8 \qquad (used\ quotient\ property)$$

$$= 0 - 8.log_{10}10 \qquad (used\ log_a 1 = 0)$$

$$= -8 \qquad (used\ log_a a = 1)$$

which is required solution.

Q.4. Change the base of $log_2 3$ to 10 i.e. common logarithm**.**

Ans. Given expression is $log_2 3$

$$= \frac{log_{10}(3)}{log_{10}(2)}$$

Q.5. Solve the equation $log_5 a^2 = 1$ for a.

Ans. Given equation is

$$log_5 a^2 = 1$$

$\Rightarrow$ $$log_5 a^2 = log_5 5$$

$\Rightarrow$ $$a^2 = 5$$

$\Rightarrow$ $$a = \pm\sqrt{5}$$

Q.6. Prove that $2log_2 4 + log_2 9 - log_2 6 = log_2 24$.

Ans. $2log_2 4 + log_2 9 - log_2 6$

$$= log_2 4^2 + log_2 9 - log_2 6$$

$$= log_2 16 + log_2 9 - log_2 6$$

$$= log_2(16 \times 9) - log_2 6$$

$$= log_2\left(\frac{16\times 9}{6}\right)$$

$$= log_2 24$$

Hence proved.